Caution: It's very possible these solutions contain mistakes or are not the most efficient.

1. (a) It is. It will be enough to verify that $R$ is a vector space following from this. First, for any partition $P$ of $[a, b]$ and $\lambda \in R$ we have $U_\alpha(P, \lambda f) - L_\alpha(P, \lambda f) = \lambda(U_\alpha(P, f) - L_\alpha(P, f))$. This follows from the equality
\[
\sup_{a,b \in I} \{\lambda f(a) - \lambda f(b)\} = \lambda \sup_{a,b \in I} \{f(a) - f(b)\}.
\]

Thus, $\lambda f$ is integrable if $f$ is. Now suppose $f, g \in R_\alpha[a, b]$ and fix $\epsilon > 0$. As $f, g \in R_\alpha[a, b]$ there exists partitions $P_1$ and $P_2$ such that $U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2}$ and $U(P_2, g) - L(P_2, g) < \frac{\epsilon}{2}$. Let $P$ be the common refinement of $P_1$ and $P_2$. We have $U(P, f + g) \leq U(P_1, f) + U(P_2, g)$ and $L(P, f + g) \geq L(P_1, f) + L(P_2, g)$. Thus $U(P, f + g) - L(P, f + g) < \epsilon$ and as $\epsilon$ was arbitrary this shows $f + g$ is integrable.

(b) It is. Suppose $f \in R_\alpha[a, b]$ and fix $\epsilon > 0$. As $f$ is integrable there exists some partition $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ such that $U_\alpha(P, f) - L_\alpha(P, f) < \epsilon$. Let $M_i(f)$ and $m_i(f)$ denote $\sup_{c \in [x_{i-1}, x_i]} \{f(c)\}$ and $\inf_{c \in [x_{i-1}, x_i]} \{f(c)\}$ respectively. We may rearrange $U_\alpha(P, |f|) - L_\alpha(P, |f|)$ as
\[
\sum (\alpha(x_{k+1}) - \alpha(x_k))(M_{k+1}|f| - m_{k+1}|f|).
\]

Now, the inequality
\[
M_k(f) - m_k(f) = \sup_{a,b \in [x_k, x_{k+1}]} \{|f(a)| - |f(b)|\} \geq \sup_{a \in [x_k, x_{k+1}]} \{|f(a)|\} - \inf_{b \in [x_k, x_{k+1}]} \{|f(b)|\} = M_k(|f|) - m_k(|f|),
\]
shows that $U_\alpha(P, |f|) - L_\alpha(P, |f|) \leq U_\alpha(P, f) - L_\alpha(P, f)$ and so $|f| \in R_\alpha[a, b]$.

(c) It is. We will first show that $f \in R_\alpha[a, b]$ implies $f^2 \in R_\alpha[a, b]$. Let $K = \|f\|_\infty < \infty$. Fix a partition $P$ such that $U_\alpha(P, |f|) - L_\alpha(P, |f|) < \frac{\epsilon}{2K}$. On any interval
[x_{k-1}, x_k] of the partition, we have the inequality
\[ M_k(f^2) - m_k(f^2) = \sup_{a,b \in [x_{k-1}, x_k]} \{ f^2(a) - f^2(b) \} \]
\[ \geq (M_k(|f|) - m_k(|f|))(M_k(|f|) + m_k(|f|)) \]
\[ \leq 2K(M_k(|f|) - m_k(|f|)), \]
which shows that \( U_\alpha(P, f^2) - L_\alpha(P, f^2) < \epsilon \) and so \( f^2 \in R_\alpha[a, b] \). Now, as \( R_\alpha[a, b] \) is a vector space, thus closed under addition and scalar multiplication, for any \( f, g \in R_\alpha[a, b] \) we have
\[ \frac{1}{2}[(f + g)^2 - f^2 - g^2] = fg \in R_\alpha[a, b]. \]

2. (a) Fix \( \epsilon > 0 \). We wish to show there exists some partition \( P \) such that \( U_\alpha(f, P) - L_\alpha(f, P) < \epsilon \). As \( f \) is continuous at each of the \( x_i \), pick \( \delta \) such that \( |f(x) - f(x_i)| < \epsilon \) if \( |x - x_i| < \delta \) for all \( 0 \leq i \leq n \). Let \( P \) be a partition \( \{a = y_0 < y_1 < ... < y_n = b\} \) such that \( y_{i+1} - y_i < \delta \) for all \( i \) and such that each interval \([y_i, y_{i+1}]\) contains at most one \( x_k \). Let \( A \subset P \) denote the set of intervals which contain some \( x_k \). We have:
\[ U_\alpha(f, P) - L_\alpha(f, P) = \sum_{a \in A} (M_k(f) - m_k(f))(\alpha(y_{k+1}) - \alpha(y_k)) \]
\[ = \sum_{a \in A} (M_k(f) - m_k(f))(\alpha(y_{k+1}) - \alpha(y_k)) \]
\[ \leq 2\epsilon \sum_{a \in A} (\alpha(y_{k+1}) - \alpha(y_k)) \]
by choice of our \( \delta \) and because \( \alpha(y_{k+1}) - \alpha(y_k) \) is 0 unless the interval \([y_k, y_{k+1}]\) contains some \( x_i \). Thus, the above sum is bounded by \( 2\epsilon \sum_{i=0}^n \alpha_i \cdot f(x_i) \leq U_\alpha(f, P) \) for all \( P \),
\[ \int_a^b f(x)d\alpha = \sum_{i=0}^n \alpha_i f(x_i). \]

(b) From (a) and because \([x]\) has jumps of size 1 we have \( \int_1^n f(x)d[x] = \sum_{k=2}^n f(k) \).
If \( t \) is not an integer then \( \int_1^t f(x)d[x] = \int_1^{[t]} f(x)d[x] = \sum_{k=2}^{[t]} f(k) \) as the function \([x]\) is constant on the interval \([t], t]. \]
(c) We will show \( \int_a^b f d\alpha = \int_a^b f \alpha' dx \). Fix \( \epsilon > 0 \). It will be enough to show that we can find partitions \( P_1 \) and \( P_2 \) such that \( U_\alpha(f, P_1) < L(f \alpha', P_1) + \epsilon \) and \( U(f \alpha', P_2) < L_\alpha(f, P_2) + \epsilon \). For any partition \( P \) of \([a, b]\),
\[ U_\alpha(f, P) - L(f \alpha', P) = \sum \left( \frac{M(f)}{x_{k+1} - x_k}(\alpha(x_{k+1}) - \alpha(x_k)(x_{k+1} - x_k)) - (m(f \alpha')(x_{k+1} - x_k)) \right) \]
\[ = \sum (x_{k+1} - x_k) \left( \frac{M(f)}{x_{k+1} - x_k}(\alpha(x_{k+1}) - \alpha(x_k)) - (m(f \alpha')) \right) \]
As $f$ and $\alpha'$ are uniformly continuous on $[a,b]$ and because $\frac{\alpha(x_{k+1}) - \alpha(x_k)}{x_{k+1} - x_k} \to \alpha'(x_k)$ as $x_{k+1} - x_k \to 0$, we can find some $\delta > 0$ such that if $P_1 = \{a = x_0 < x_1 < ... < x_n = b\}$ is a partition with $x_{k+1} - x_k < \delta$ for all $i$, then

$$M(f) \frac{\alpha(x_{k+1}) - \alpha(x_k)}{x_{k+1} - x_k} - m(f) < \frac{\epsilon}{b - a}$$

for all $i$. Thus,

$$U_\alpha(f, P_1) - L(f, P_1) \leq \frac{\epsilon}{b - a} \sum_{i=1}^n (x_{k+1} - x_k) = \epsilon.$$

With a similar computation to the above we can also find a partition $P_2$ with $U(f, P_2) < L_\alpha(f, P_2) + \epsilon$, which gives us our result.

3. (a) FALSE. Every interval of positive length contains both rational and irrational numbers. Thus for any partition $P$ of $[0,1]$ we have $U(\chi_Q, P) = 1$ and $L(\chi_Q, P) = 0$ and so $\chi_Q$ is not integrable.

(b) TRUE. Let $C_0$ denote the unit interval and for $k \geq 1$ let $C_k$ the union of intervals obtained from $C_{k-1}$ by deleting the middle third of intervals making up $C_{k-1}$. We know $C = \bigcap_0^\infty C_k$. We also know $\int_0^1 \chi_{C_k} dx = (\frac{2}{3})^k$. For all $k$, $0 \leq \int_0^1 \chi_{C_k} dx \leq \int_0^1 \chi_C dx = 0$ and $\chi_C$ is integrable.

(c) TRUE. We already know $\bigcap_0 \{R_\alpha[a,b] \} \supseteq C([a,b], \mathbb{R})$ so we show the other inclusion. Suppose $f$ is left discontinuous at $c \in [a,b]$ and let $\alpha = \chi_{[c,b]}$. Let $A$ denote the set of intervals in $[a,b]$ containing $c$ and for which $c$ is not the left end-point. As $f$ is left discontinuous at $c$, $\inf_{I \in A} \{M_I(f) - m_I(f)\} = d > 0$. For any partition $P = \{a = x_0 < ... < b = x_n\}$ of $[a,b]$ let $I = [x_{k-1}, x_k] \in A$ be interval containing $c$ and for which $c$ is not the left end-point. We have

$$U_\alpha(f, P) - L_\alpha(f, P) \geq \left( M_k(f) - m_k(f) \right) \chi_{[c,b]}(x_k) - \chi_{[c,b]}(x_{k-1}) \geq d > 0.$$

As this is true for all partitions $P$, $f \notin R_\alpha[a,b]$. A similar proof works if $f$ is right discontinuous.

(d) TRUE. As both $f$ and $\alpha$ are non-decreasing, on any interval $[x_{i-1}, x_i]$ we have $M_i(f) = f(x_i)$, $m_i(f) = f(x_{i-1})$, $M_i(\alpha) = \alpha(x_i)$ and $m_i(\alpha) = \alpha(x_{i-1})$. Hence, for any partition $P$ of $[a,b]$ we have $U_\alpha(f, P) = \sum_{i=1}^n (M_i(f) - m_i(f))(M_i(\alpha) - m_i(\alpha)) = U_f(\alpha, P)$ and similarly $L_\alpha(f, P) = L_f(\alpha, P)$. As $\alpha$ is continuous and $f$ is monotone we know $\alpha \in R_f[a,b]$ and from the above observation we also have $f \in R_\alpha[a,b]$.

(e) FALSE. Suppose both $f$ and $\alpha$ are left discontinuous at $c \in [a,b]$. Let $A$ denote the set of intervals in $[a,b]$ containing $c$ and for which $c$ is not the left end-point. As $f$ and $\alpha$ are left discontinuous at $c$, $\inf_{I \in A} \{M_I(f) - m_I(f)\} = d_1 > 0$ and $\inf_{I \in A} \{M_I(\alpha) - m_I(\alpha)\} = d_2 > 0$. For any partition $P$,

$$U_\alpha(f, P) - L_\alpha(f, P) \geq \left( M_k(f) - m_k(f) \right) \left( \alpha(x_k) - \alpha(x_{k-1}) \right) \geq d_1 d_2 > 0.$$

As this is true for all partitions $P$, $f \notin R_\alpha[a,b]$. A similar proof works if $f$ and $\alpha$ are right discontinuous.
(f) TRUE. Proven in theorem 6.11 of Rudin

(g) TRUE. We will first show that $\| \cdot \|_{BV}$ defines a norm on $BV[a, b]$. We have $\|f\|_{BV} = 0 \iff f = 0$ as only constant functions have zero variation. Next, $\|\lambda f\|_{BV} = |\lambda f(a)| + V_a^b(f) = \lambda |f(a)| + \lambda V_a^b(f) = \lambda \|f\|_{BV}$. To see the triangle inequality holds let $f, g \in BV[a, b]$. We only need to show $V_a^b(f + g) \leq V_a^b(f) + V_a^b(g)$ as we know $|(f + g)(a)| \leq |f(a)| + |g(a)|$ by triangle inequality on $\mathbb{R}$. For any partition $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of $[a, b]$, by the triangle inequality on $\mathbb{R}$,

$$V_a^b((f + g), P) = \sum |(f + g)(x_k) - (f + g)(x_{k-1})|$$

$$\leq \sum |f(x_k) - f(x_{k-1})| + \sum |g(x_k) - g(x_{k-1})|$$

$$= V_a^b(f, P) + V_a^b(g, P).$$

Taking the sup over partitions $P$ we get $V_a^b(f + g) \leq V_a^b(f) + V_a^b(g)$. We now demonstrate completeness. Suppose $\{f_n\}$ is a Cauchy sequence in $BV[a, b]$. Note that it will also be a Cauchy sequence in the sup norm as for any $f$ in $BV[a, b]$,

$$\|f\|_{BV} = f(a) + V_a^b(f) \geq f(a) + \sup_{x \in [a, b]} |f(x) - f(a)| \geq \sup_{x \in [a, b]} |f(x)| = \|f\|_\infty.$$

By completeness of the space $B[a, b]$ of bounded functions on $[a, b]$, the Cauchy sequence $\{f_n\}$ converges pointwise to some bounded function $f$. It remains to show that $f$ is of bounded variation. Pick $f_k \in \{f_n\}$ such that $\|f_k - f\|_{BV} < 1$. By the triangle inequality,

$$V_a^b(f) \leq V_a^b(f - f_k) + V_a^b(f_k) < 1 + V_a^b(f_k) < \infty$$

and $f$ is of bounded variation.