Caution: It’s very possible these solutions contain mistakes or are not the most efficient.

1. Let \( \{r_i\} \) be an enumeration of the rationals in \([0, 1]\). We successively define subsequences \( \{f_k^i\} \) as follows. Let \( a_1 \) be a limit point of \( \{f_n(r_1)\} \) (this will exist as \([0, 1]\) is compact) and pick a subsequence \( \{f_{n_k}(r_1)\} \) of \( \{f_n\} \) such that \( |f_{n_k}(r_1) - a_1| \leq \frac{1}{k} \) for all \( k \geq 1 \).

Now suppose we have already chosen the subsequence \( \{f_{i_k}^j\} \), we define the subsequence \( \{f_{i_k+1}^j\} \) of \( \{f_{i_k}^j\} \) as follows. Let \( a_{i+1} \) be a limit point of \( \{f_{i_k}^j(r_{i+1})\} \) and pick \( \{f_{i_k+1}^j\} \) such that \( |f_{i_k+1}^j(r_{i+1}) - a_{i+1}| \leq \frac{1}{k} \) for all \( k \geq 1 \). Let \( \{g_k\} \) be the diagonal of these subsequences, that is, \( g_j := f_j^j \) for all \( j \geq 1 \). For all \( r_i \in [0, 1] \cap \mathbb{Q} \) we have \( \lim_{n \to \infty} g_n(r_i) = a_i \) as desired.

2. No it does not. Let \( f_n(x) = \frac{x}{n} \). On any interval of the form \([-a, a] \) with \( a > 0 \) we have \( |f_n(x)| \leq \frac{a}{n} \). Uniform convergence to 0 on compact subsets of \( \mathbb{R} \) follows immediately from this as any such set is contained in some \([-a, a] \). However, \( f_n \) is not uniformly convergent on \( \mathbb{R} \) as for all \( n \) we have \( f_n(2n) = 2 > 1 \).

3. (a) We have \( f_n(0) = f_n(1) = 0 \) and \( \lim_{n \to \infty} n^2 x(1 - x^2)^n = 0 \) for all \( x \in (0, 1) \). Hence, the pointwise limit of \( f_n \) is 0 on \([0, 1]\). We now show that \( f_n \) will be uniformly convergent exactly on intervals contained in ones of the form \([a, 1]\), with \( a > 0 \). To see it can’t converge uniformly on intervals of the form \([0, b]\) define a sequence of points \( \{x_n\} \) with \( x_n := \frac{1}{\sqrt{n}} \). Computing,

\[
\lim_{n \to \infty} f_n(x_n) = \lim_{n \to \infty} n^{3/2} e^{-1} \to \infty,
\]

so uniform convergence on \([0, b]\) is impossible. To see uniform convergence on \([a, 1]\), it can be verified that for large \( n \), \( f_n \) achieves its maximum on \([a, 1]\) at \( a \). As

\[
\lim_{n \to \infty} f_n(a) = \lim_{n \to \infty} n^2 a(1 - a^2)^n \to 0
\]

we have uniform convergence on \([a, 1]\).

(b) Using calculus, we find that the maximum of \( xe^{-nx} \) occurs at \( x = \frac{1}{n} \). As

\[
\lim_{n \to \infty} \left( \sup_{x \geq 0} |xe^{-nx}| \right) = \lim_{n \to \infty} \frac{1}{n} e^{-1} \to 0
\]

we see that the \( f_n \) converge uniformly to 0 on \( \mathbb{R}_+ \) and so 0 is the pointwise limit as well.
4. (a) \( \Rightarrow \) If \( \sum_1^\infty 2^{-n} \frac{d_n(f_k,f)}{1 + d_n(f_k,f)} \xrightarrow{k} 0 \) then for each \( n \) we must have \( d_n(f_k,f) \xrightarrow{k} 0 \) as all the terms in the sum are positive and because \( \frac{d_n(f_k,f)}{1 + d_n(f_k,f)} \to 0 \) iff \( d_n(f_k,f) \xrightarrow{k} 0 \). From the definition this means \( f_n \) converges uniformly to \( f \) on \([-n,n]\) and any compact interval is contained in something of this form.

\( \Leftarrow \) Fix \( \epsilon > 0 \). Let \( n_\epsilon \) be chosen such that \( \sum_{n_\epsilon}^\infty 2^{-k} < \frac{\epsilon}{2} \). We then have

\[
\sum_1^\infty 2^{-n} \frac{d_n(f_k,f)}{1 + d_n(f_k,f)} = \sum_1^{n_\epsilon} 2^{-k} \frac{d_n(f_k,f)}{1 + d_n(f_k,f)} + \sum_{n_\epsilon}^\infty 2^{-k} \frac{d_n(f_k,f)}{1 + d_n(f_k,f)} \\
\leq \sum_1^{n_\epsilon} 2^{-k} \frac{d_n(f_k,f)}{1 + d_n(f_k,f)} + \frac{\epsilon}{2} \\
\leq \sum_1^{n_\epsilon} 2^{-k} \frac{d_{n_\epsilon}(f_k,f)}{1 + d_{n_\epsilon}(f_k,f)} + \frac{\epsilon}{2} \\
\leq d_{n_\epsilon}(f_k,f) + \frac{\epsilon}{2}
\]

Here, the second inequality follows from the fact that \( d_{n_1}(h_1,h_2) \leq d_{n_2}(h_1,h_2) \) for \( n_1 \leq n_2 \). By uniform convergence on the interval \([-n_\epsilon,n_\epsilon]\) we can find \( n \) such that \( d_{n_\epsilon}(f_k,f) \leq \frac{\epsilon}{2} \) for all \( k \geq n \).

(b) Let \( \{f_n\} \) be a Cauchy sequence in \( C(\mathbb{R}) \). If \( d(f_n,f_m) \xrightarrow{\text{m,n}} 0 \) then we must have \( d_l(f_n,f_m) \xrightarrow{\text{m,n}} 0 \) for all \( l > 0 \). From the definition of \( d_l \), this implies that for all \( x \in [-l,l] \), the sequence \( \{f_n(x)\} \) is Cauchy and by completeness of \( \mathbb{R} \) it converges to some \( a_x \in \mathbb{R} \). Hence, \( \{f_n\} \) converges pointwise to some function \( f \) on any compact interval and therefore converges pointwise to \( f \) on all of \( \mathbb{R} \). We know that the convergence of \( \{f_n\} \) to \( f \) will be uniform on compact intervals as

\[
\lim_{n \to \infty} \sup_{x \in [-l,l]} |f_n(x) - f(x)| = \lim_{n \to \infty} \sup_{x \in [-l,l]} |f_n(x) - \lim_{k \to \infty} f_k(x)| = 0
\]

where the last equality holds as \( d_l(f_n,f_m) \xrightarrow{\text{m,n}} 0 \). As a sequence of continuous functions uniformly converges to \( f \) on \([-l,l] \), \( f \) must itself be continuous on \([-l,l] \).

As continuity is a local property, taking \( l \to \infty \) we see \( f \) is continuous everywhere.

5. We will show that \( g_n \to 0 \) uniformly, where \( g_n(x) := \sum_n^{\infty} a_k \sin kx \). Fix \( \epsilon > 0 \) and pick \( N \) such that \( \left| \sum_N^{\infty} a_k \right| < \epsilon \). As \( \sin \) is bounded by 1, we obtain that \( |g_N(x)| < \epsilon \) for all \( x \). This proof carries over for \( a_n \cos nx \).