HW11 solutions

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1. (a) We claim the range of \( f(x, y) = (e^x \cos y, e^x \sin y) \) is \( \mathbb{R}^2 \setminus (0, 0) \). Indeed, any \( x = \mathbb{R}^2 \setminus (0, 0) \) may be written in polar coordinates as \( (r \cos \theta, r \sin \theta) \), with \( r > 0 \). Then \( f(\log(r), \theta) = x \).

(b) We compute the Jacobian of \( f \) to be

\[
\begin{bmatrix}
e^x \cos y & -e^x \sin y \\
e^x \sin y & e^x \cos y
\end{bmatrix},
\]

which has determinant \( e^{2x} \neq 0 \). Hence, by the inverse function theorem \( f \) is locally invertible on its domain. However, \( f \) is \( 2\pi \)-periodic with respect to \( x \) and therefore not globally invertible.

(c) Let \( U \) be the infinite strip \( U = \{(x, y) | x \in (-\frac{\pi}{2}, \frac{\pi}{2}), y \in \mathbb{R}\} \). For \( (x, y) \in U \), we write \( f(x, y) = (e^x \cos y, e^x \sin y) = (u, v) \). This gives us \( x = \log(u^2 + v^2) \) and \( y = \arctan(\frac{v}{u}) \). Hence on \( U \), the inverse of \( f \) is given by \( g(u, v) = (\log(u^2 + v^2), \arctan(\frac{v}{u})) \). If \( a = (0, \frac{\pi}{3}) \), then \( f(a) = (\frac{1}{2}, \frac{\sqrt{3}}{2}) \). By the inverse function theorem, \( [J_g(b)] = [J_f(a)]^{-1} \), thus:

\[
J_g(b) = \begin{bmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix}^{-1} = \begin{bmatrix}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
-\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix}.
\]

2. Consider the system the system

\[
\begin{align*}
3x + y - z + u^2 &= 0 \quad (1) \\
x - y + 2z + u &= 0 \quad (2) \\
2x + 2y - 3z + 2u &= 0. \quad (3)
\end{align*}
\]

The difference \( (1) - (2) - (3) \) gives us \( u^2 - 3u = 0 \). Hence, any solution to the above system must have \( u = 0 \) or \( u = 3 \). This furthermore tells us it is not possible to solve for \( u \) in terms of \( x, y, z \). Now, showing the system can be solved say for \( y, z, u \in \mathbb{R} \) in terms of \( x \) is equivalent to showing the zero set of the function \( f : \mathbb{R}^4 \to \mathbb{R}^3 \) given by \( f(x, y, z, u) = (3x + y - z + u^2, x - y + 2z + u, 2x + 2y - 3z + 2u) \) can be written as the image of a curve in \( \mathbb{R}^4 \) of the form...
(t, g_1(t), g_2(t), g_3(t)) with g_1 : \mathbb{R} \rightarrow \mathbb{R}. From the Implicit Function Theorem, to demonstrate the existence of such a curve it is enough to show that the determinant of the matrix

\[
[Df_y, Df_z, Df_u] = \begin{bmatrix}
1 & -1 & 2u \\
-1 & 2 & -3 \\
0 & 1 & 2
\end{bmatrix}
\]

is non-zero. We compute the determinant of this matrix to be 3 - 2u, which is non-zero for u = 0, 3. A similar argument will show that it is possible to solve for y in terms of x, z, u and solve for z in terms of y, z, u.

3. Let f(x, y_1, y_2) = x^2 y_1 + e^x + y_2. By the implicit function theorem, the existence of a function g : \mathbb{R}^2 \rightarrow \mathbb{R} on some neighbourhood U \subset \mathbb{R}^2 of (−1, 1) such that f(g(y_1, y_2), y_1, y_2) = 0 for (y_1, y_2) ∈ U and such that g(−1, 1) = 0 follows because f(0, −1, 1) = 0 and because the 1 × 1 matrix \[Df_x(0, −1, 1)\] = Df_x is non-zero. We now compute the partials of g. We have \[f(g(y_1, y_2), y_1, y_2) = g^2 y_1 + e^g + y_2 = 0\]. Differentiating both sides with respect to y_1 we obtain 2gg_y_1 + g^2 + e^g_y_1 = 0, hence \[g_y_1(−1, 1) = 0\]. Similarly, differentiating with respect to y_2 we obtain 2gg_y_2 + e^g_y_2 + 1 = 0 and so \[g_y_2(−1, 1) = −1\].

4. (a) Pick an arbitrary \(x_0 \in M\). As \(f\) is a contraction mapping, for all \(n \geq 2\) we have \(d(f^n(x_0), f^{n−1}(x_0)) \leq \alpha d(f^{n−2}(x_0), f^{n−1}(x_0))\) (here \(f^n\) will denote the \(n\)-fold composition of \(f\) with itself). Hence, \(d(f^n(x_0), f^{n−1}(x_0)) \leq \alpha^{n−1} d(f(x_0), x_0)\). As \(\alpha < 1\) we obtain \(\lim_{n \to \infty} d(f^n(x_0), f^{n−1}(x_0)) = 0\) and by completeness of \(M\), the Cauchy sequence \(\{f^n(x_0)\}\) converges to some element \(x\) of \(M\). We indeed have \(f(x) = x\) as \(f(x) = f(\lim_{n \to \infty} f^n(x_0)) = \lim_{n \to \infty} f^{n+1}(x_0) = x\). To see the uniqueness of the fixed point suppose we also have \(y \in M\) such that \(d(f(y), y) = 0\). As \(x\) and \(y\) are fixed points, \(d(f(y), f(x)) = d(y, x)\). But if \(x \neq y\) then \(d(f(y), f(x)) < d(y, x)\) as \(f\) is a contraction map, so we must have \(x = y\).

(b) We will first consider the fixed point at \(p_0 = 0\). Let \(\delta = \frac{1}{2}\). Then as \(|f'| < 1\) on \((-\frac{1}{2}, \frac{1}{2})\) by the Mean Value Theorem we see \(|f(x) - f(p_0)| < |x - p_0|\) and hence \(f\) is a contraction mapping on that interval. From part a) we know the fixed point \(p_0 = 0\) will be unique and furthermore \(f^n(x) → p_0\) for all \(x \in (-\frac{1}{2}, \frac{1}{2})\).

Now we consider the fixed point \(p_1 = 1\) and let \(\delta = \frac{1}{2}\). As \(f' > 1\), for every \(x \in (\frac{1}{2}, \frac{3}{2})\) with \(x \neq p_1\), again by the Mean Value Theorem theorem we have \(|f(x) - p_0| > |x - p_0|\). We have \(f^n(x) = x^{2^n}\), hence for \(|x| > 1\), \(f^n(x)\) diverges and for \(|x| < 1\), \(f^n(x) → 0\).

(c) If \(|f'(p)| < 1 - \epsilon\) for some \(\epsilon > 0\), then by definition of differentiability there exists some \(\delta\) such that \(\frac{f(h+p)-f(p)}{h} < 1 - \frac{\epsilon}{2}\) for all \(h \in (-\delta + p, \delta + p)\). For all \(x \in (-\delta + p, \delta + p)\) we then have \(|f(x) - f(p)| < |x - p|\) by MVT and so from part b) we see that \(p\) will be an attracting fixed point.

Likewise, if \(|f'(p)| > 1 - \epsilon\) for some \(\epsilon > 0\), then by definition of differentiability there exists some \(\delta\) such that \(\frac{f(h+p)-f(p)}{h} > 1 - \frac{\epsilon}{2}\) for all \(h \in (-\delta + p, \delta + p)\). For all \(x \in (-\delta + p, \delta + p)\) we then have \(|f(x) - f(p)| > |x - p|\) and so from part b) we see that \(p\) will be a repelling fixed point.

(d) i. Let \(f_1(x) = \arctan(x)\) and \(p_1 = 0\). Observe that on \((-1, 1)\) we have \(f_1' \leq 1\) with equality only at 0. By the Mean Value Theorem, \(|f(x) - f(p_1)| < |x - p_1|\) for \(x \in (-1, 1)\) and from part b) we know \(p_1\) will be an attracting fixed point.
ii. Let $f_2(x) = x^3 + x$ and $p_2 = 0$. Observe that on $\mathbb{R}$ we have $f_2' \geq 1$ with equality only at 0. By the Mean Value Theorem, $|f(x) - f(p_2)| > |x - p_2|$ for $x \in \mathbb{R}$ and from part b) we know $p_2$ will be an repelling fixed point.

iii. Let $f_2(x) = x^2 + \frac{1}{4}$ and $p_3 = \frac{1}{2}$. Observe that for $x \in (\frac{1}{2}, +\infty)$ we have $|f(x) - f(p_2)| > |x - p_2|$ and for $x \in (0, \frac{1}{2})$ we have $|f(x) - f(p_2)| < |x - p_2|$. Thus for $x \in (\frac{1}{2}, +\infty)$, $f_3^n(x) \to +\infty$ and for $x \in (0, \frac{1}{2})$, $f_3^n(x) \to p_3$. This tells us $p_3$ is neither an attracting fixed point nor a repelling fixed point.

(e) Let $f(x) = x^3 - x - 1$. As $f(1) = -1 < 0$ and $f(2) = 5 > 0$, by IVT $f$ will have a root $x_0$ in $(1, 2)$. This root will be unique as $f$ is 1-1 on $(1, 2)$ because $f'(x) = 3x^2 - 1 > 0$ there. Now observe that the fixed points of $\tilde{f}(x) = \sqrt[3]{x+1}$ will exactly be the roots of $f(x)$ as if $\tilde{f}(x_0) = x_0$ then $x_0 + 1 = x_0^3$. As $|\tilde{f}'| < 1$ on $(1, 2)$, part b) tells us $x_0$ is an attracting fixed point. Thus, for any $x \in (1, 2)$ we have $\lim_{n \to \infty} \tilde{f}^n(x) = x_0$ as desired.