1. Let \( S \) denote the set of functions in \( C[-\pi, \pi] \) of the form
\[
f(x) = a \sin x + b \sin 2x
\]
where \( a \) and \( b \) are arbitrary real numbers. Let \( g(x) = x \) for \( x \in [-\pi, \pi] \). Find \( f \in S \) for which \( ||g - f||_2 \) is smallest.

\((Answer: \ f(x) = 2 \sin x - \sin 2x.)\)

2. Let \( f: [0,1] \times [0,1] \to \mathbb{R} \) be the function
\[
f(x,y) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q}, \\
2y & \text{if } x \notin \mathbb{Q}.
\end{cases}
\]

(a) Compute the lower and upper Riemann integrals
\[
\int_0^1 f(x,y) \, dx \quad \text{and} \quad \int_0^1 f(x,y) \, dx
\]
in terms of \( y \).

(b) Show that
\[
\int_0^1 f(x,y) \, dy \quad \text{exists for each fixed } x.
\]
Compute
\[
\int_0^t f(x,y) \, dy \quad \text{in terms of } (x,t) \in [0,1] \times [0,1].
\]

(c) Define
\[
F(x) = \int_0^1 f(x,y) \, dy.
\]
Show that \( \int_0^1 F(x) \, dx \) exists and find its value.

(d) There must be a moral to this long-winded story. What is it?

3. A certain Riemann-integrable function \( f: [-\pi, \pi] \to \mathbb{C} \) and a complex sequence \( \{c_k\} \) obey
\[
||f(t) - \sum_{k=-n}^{n} c_k e^{ikt}||_2 \to 0 \quad \text{as} \quad n \to \infty.
\]
Prove the following statements:

(a) For any \( g: [-\pi, \pi] \to \mathbb{C} \) with \( g \in \mathcal{R}[-\pi, \pi] \),
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \overline{g(t)} \, dt = \sum_{k=-\infty}^{\infty} c_k \hat{g}(k), \quad \text{where} \quad \hat{g}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-ikt} \, dt.
\]

(b) \( c_k = \hat{f}(k) \) and \( \sum_k |c_k|^2 < \infty. \)
4. Evaluate the following, with careful justification of all steps:

\[
\sum_{n=-\infty}^{\infty} \left| \int_{-\pi}^{\pi} t^5 e^{-int} dt \right|^2
\]

(Answer: \( \frac{4\pi^{12}}{11} \).

5. Let \( g : [0, 1] \to \mathbb{R} \) be bounded and \( \alpha : [0, 1] \to \mathbb{R} \) be nondecreasing. Assume that \( g \in \mathcal{R}_\alpha[\delta, 1] \) for every \( \delta > 0 \).

(a) Show that \( g \in \mathcal{R}_\alpha[0, 1] \) if \( \alpha \) is continuous at 0.

(b) Give an example of a pair \((g, \alpha)\) which shows that the conclusion of part (a) is false if \( \alpha \) is not assumed to be continuous at 0.

6. Let

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

be the Fourier series of a function \( f \in \text{BV}[-\pi, \pi] \). Show that \( \{na_n\} \) and \( \{nb_n\} \) are bounded sequences.

7. Determine whether or not the following functions \( f \) are of bounded variation on \([0, 1]\).

(a) \( f(x) = x^2 \sin(\frac{1}{x}) \) if \( x \neq 0 \), \( f(0) = 0 \).

(Answer: Yes.)

(b) \( f(x) = \sqrt{x} \sin(\frac{1}{x}) \) if \( x \neq 0 \), \( f(0) = 0 \).

(Answer: No.)

8. A function \( f : [a, b] \to \mathbb{R} \) is said to satisfy a Lipschitz or Hölder condition of order \( \alpha > 0 \) if there exists \( M > 0 \) such that

\[
|f(x) - f(y)| < M|x - y|^{\alpha} \text{ for all } x, y \in [a, b].
\]

(a) If \( f \) is such a function, show that \( \alpha > 1 \) implies that \( f \) is constant on \([a, b] \), whereas \( \alpha = 1 \) implies \( f \in \text{BV}[a, b] \).

(b) Give an example of a function not of bounded variation satisfying a Hölder condition of order \( \alpha < 1 \).

(c) Give an example of a function of bounded variation on \([a, b] \) that satisfies no Lipschitz condition on \([a, b] \).