Math 321 Midterm 2 Solutions

1. (a) When is a function \( \alpha : [a, b] \to \mathbb{R} \) said to be of bounded variation?

Solution. A function \( \alpha : [a, b] \to \mathbb{R} \) is said to be of bounded variation if its total variation \( V^b_a \alpha \) is finite. The total variation is defined to be

\[
V^b_a \alpha = \sup \sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})|,
\]

where the supremum is taken over all partitions \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \) of \([a, b]\). □

(b) Determine whether the function \( \alpha : [0, 1] \to \mathbb{R} \) given by

\[
\alpha(x) = \begin{cases} 
\log(1 + x) \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0, \\
0 & \text{if } x = 0,
\end{cases}
\]

is of bounded variation.

Solution. The given function \( \alpha \) is not of bounded variation. To see this, let us choose, for every large integer \( N \), a partition \( P_N \) of \([0, 1]\) of the form

\[
P_N = \{1 = t_0 > t_1 > t_2 > \cdots t_{2N} > t_{2N+1} = 0\}, \text{ where } t_k = \frac{2}{k\pi}, 1 \leq k \leq 2N.
\]

Since \( \sin(1/t_k) \) vanishes for even \( k \) and equals \( \pm 1 \) for odd \( k \), one of the terms in any pair \((\alpha(t_k), \alpha(t_{k+1}))\) must vanish. This means that

\[
|\alpha(t_k) - \alpha(t_{k+1})| = \log(1 + s_k) \text{ where } s_k = \begin{cases} 
t_k & \text{if } k \text{ is odd}, \\
t_{k+1} & \text{if } k \text{ is even}.
\end{cases}
\]

In other words,

\[
(1) \quad \sum_{k=1}^{N} |\alpha(t_k) - \alpha(t_{k-1})| \geq \sum_{k=1}^{N} \log(1 + t_{2k-1}) = \sum_{k=1}^{N} \log \left( 1 + \frac{2}{\pi(2k-1)} \right).
\]

We know that

\[
\frac{\log(1 + x)}{x} \to 1 \text{ as } x \to 0.
\]

Therefore by limit comparison test, the last series in (1) is comparable to the partial sum \( \sum_{k=1}^{N} 1/k \) of the harmonic series, which diverges to \( \infty \) as \( N \to \infty \). □

(c) A linear functional \( L : C[0, 1] \to \mathbb{R} \) obeys the following property: for every continuously differentiable \( g : [0, 1] \to \mathbb{R} \),

\[
L(g) = -\int_{0}^{1} g'(x) \cos(\pi x) \, dx.
\]

Does there exist \( \alpha \in BV[0, 1] \) such that

\[
L(f) = \int_{0}^{1} f(x) d\alpha(x), \text{ for every } f \in C[0, 1]?
\]

If yes, find such a function \( \alpha \). If not, explain why not. Clearly state any result you need to use.
Solution. Integrating by parts, we find that for every continuously differentiable function \( g \),
\[
L(g) = -\cos(\pi x)g(x) \bigg|_{x=0}^{x=1} + \int_0^1 (-\pi) \sin(\pi x)g(x) \, dx
\]
\[
= g(1) + g(0) - \pi \int_0^1 g(x) \sin(\pi x) \, dx
\]
\[
= g(1) + g(0) + \int_0^1 g(x) d(\cos(\pi x)).
\]
The last expression is linear in \( g \), is meaningful for every continuous function \( g \) (not merely continuously differentiable), and is bounded above in absolute value by a constant multiple of \( \|g\|_{\infty} \). Thus by the Riesz representation theorem, \( L \) is given by a Riemann-Stieltjes integral with respect to an integrator \( \alpha \in BV[0,1] \). In this case, one possible choice of \( \alpha \) is the following:
\[
\alpha(x) = \cos(\pi x) + \begin{cases} 
0 & \text{if } x = 0, \\
1 & \text{if } 0 < x < 1, \\
2 & \text{if } x = 1.
\end{cases}
\]

2. For each of the following statements, determine whether it is true or false. The answer should be in the form of a short proof or an example, as appropriate.

(a) There exists a bounded function on \([a,b]\) that fails to be Riemann-Stieltjes integrable with respect to every nondecreasing non-constant integrator \( \alpha \).

Proof. True. The function
\[
f(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q}, \\
0 & \text{if } x \notin \mathbb{Q}
\end{cases}
\]
is such a function. For any nondecreasing \( \alpha \) such that \( \alpha(a) < \alpha(b) \), and any partition \( P \) of \([a,b]\),
\[
L_\alpha(P,f) = 0, \text{ but } U_\alpha(P,f) = \alpha(b) - \alpha(a).
\]
Hence Riemann’s condition fails. \( \square \)

(b) The class \( C[a,b] \) consists of all functions that are Riemann-Stieltjes integrable on \([a,b]\) with respect to every nondecreasing integrator \( \alpha \).

Proof. True. Let \( \mathcal{R}_\alpha[a,b] \) denote the class of functions that are Riemann-Stieltjes integrable with respect to the integrator \( \alpha \). Using Riemann’s condition, we have shown in class that \( C[a,b] \subseteq \mathcal{R}_\alpha[a,b] \) for every nondecreasing \( \alpha \). Conversely, suppose \( f \) is discontinuous at a point \( x_0 \in [a,b] \). This means that
\[
\sup \{|f(x) - f(y)| : x, y \in (x_0 - \delta, x_0 + \delta)\} \to \epsilon_0 > 0 \text{ as } \delta \to 0 +.
\]
Let \( \alpha \) be a nondecreasing step function with a unit jump only at the point \( x_0 \), with the same-sided discontinuity as \( f \). Then for any sufficiently fine partition \( P \) with \( x_0 \) as a partition point, we have
\[
U_\alpha(P,f) - L_\alpha(P,f) \geq \epsilon_0,
\]
which violates Riemann’s condition. \( \square \)
(c) The Fourier series of a continuous 2π-periodic function $f$ converges to $f$ in the $L^1$ norm $\| \cdot \|_1$. Recall

$$\| f \|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| \, dx.$$  

Proof. True. By the Cauchy-Schwarz inequality, $\|g\|_1 \leq \|g\|_2$ for any Riemann-integrable function $g$. Let $S_Nf$ denote the $N$th partial Fourier sum of $f$. Since we know that $\| S_Nf - f \|_2 \to 0$ by Plancherel’s theorem, it follows from the inequality above that $\| S_Nf - f \|_1 \to 0$ as $N \to \infty$. □

(d) For any bounded, Riemann integrable function $f$, the sequence of Fourier coefficients $\{ \hat{f}(k) : k \geq 0 \}$ converges to zero.

Proof. True. Since a bounded Riemann-integrable function is square-integrable, we know by Plancherel’s theorem that

$$\sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty$$

Thus the left hand side is a summable series, and hence the $k$th summand goes to zero as $k \to \infty$. □

(e) Let $f$ be a bounded Riemann integrable function on $[-\pi, \pi]$. Then $\| \sigma_N f - f \|_1 \to 0$ as $N \to \infty$. Here $\sigma_N f$ denotes the $N$th partial Cesaro sum of $f$.

Solution. Fix $\epsilon > 0$. By HW 7 Problem 4(a) we know that there exists a continuous 2π-periodic function $g$ such that $\| f - g \|_2 < \epsilon$. By Fejer’s theorem, we know that $\| \sigma_N g - g \|_\infty \to 0$ as $N \to \infty$. Combining these steps together and using the triangle inequality, we find that

$$\| \sigma_N f - f \|_1 \leq \| \sigma_N (f - g) \|_1 + \| f - g \|_1 + \| \sigma_N g - g \|_1$$

$$\leq 2\| f - g \|_1 + \| \sigma_N g - g \|_\infty$$

$$\leq 2\epsilon + \| \sigma_N g - g \|_\infty \to 2\epsilon \text{ as } N \to \infty.$$ 

The estimate $\| \sigma_N (f - g) \|_1 \leq \| f - g \|_1$ used in the second step follows from the fact that for any function $h$, 

$$\| \sigma_N h \|_1 = \| K_N * h \|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x-y)h(y) \, dy \right| \, dx$$

$$\leq \| K_N \|_1 \| h \|_1 = \| h \|_1,$$

where $K_N$ denotes the Fejer kernel. □

3. Let $\alpha, \beta > 0$. Evaluate the sum

$$S = \sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{2\alpha} y^{2\beta} \cos(m(x+y)) \, dy \, dx.$$ 

Solution. Let us define 2π-periodic functions $f_\alpha$ and $f_\beta$ as follows:

$$f_\alpha(x) = (-x)^{2\alpha}, \quad f_\beta(x) = x^{2\beta}, \quad x \in [-\pi, \pi].$$
We first simplify $S$ as follows:

$$S = \text{Re} \left[ \sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{2\alpha} y^{2\beta} e^{im(x+y)} \, dy \, dx \right]$$

$$= \text{Re} \left[ \sum_{m \in \mathbb{Z}} \left\{ \int_{-\pi}^{\pi} x^{2\alpha} e^{imx} \, dx \right\} \left\{ \int_{-\pi}^{\pi} y^{2\beta} e^{-imy} \, dy \right\} \right]$$

$$= \text{Re} \left[ \sum_{m \in \mathbb{Z}} \left\{ \int_{-\pi}^{\pi} (-x)^{2\alpha} e^{-imx} \, dx \right\} \left\{ \int_{-\pi}^{\pi} y^{2\beta} e^{-imy} \, dy \right\} \right]$$

$$= 4\pi^2 \text{Re} \sum_{m \in \mathbb{Z}} \hat{f}_\alpha(m) \hat{f}_\beta(m)$$

$$= 4\pi^2 \text{Re} \left[ \langle \hat{f}_\alpha, \hat{f}_\beta \rangle_{\ell^2} \right] = 4\pi^2 \text{Re} \left[ \langle f_\alpha, f_\beta \rangle_{L^2} \right] = 2\pi \int_{-\pi}^{\pi} f_\alpha(x) f_\beta(x) \, dx$$

$$= \frac{2\pi^{2\alpha+2\beta+1}}{2\alpha + 2\beta + 1} \text{Re}(1 - (-1)^{2\alpha+2\beta+1}).$$

The third last inequality is a consequence of the fact that inner product is preserved under the Fourier transform. □

Remark: If we assume that $\alpha, \beta$ are positive integers, then the proof permits an additional simplification. Now the functions $f_\alpha(x) = x^{2\alpha}$, $f_\beta(x) = x^{2\beta}$ are even, and hence their Fourier series do not contain any terms involving sines. Combining this fact with the trig identity $\cos(a + b) = \cos a \cos b - \sin a \sin b$, we find that

$$S = \sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{2\alpha} y^{2\beta} \cos(mx) \cos(my) \, dy \, dx$$

$$= 4\pi^2 \sum_{m \in \mathbb{Z}} \hat{f}_\alpha(m) \hat{f}_\beta(m) = 4\pi^2 \langle \hat{f}_\alpha, \hat{f}_\beta \rangle_{\ell^2}$$

$$= 4\pi^2 \langle f_\alpha, f_\beta \rangle_{L^2} = 2\pi \int_{-\pi}^{\pi} f_\alpha(x) f_\beta(x) \, dx = \frac{4\pi^{2\alpha+2\beta+1}}{2(\alpha + \beta) + 1}.$$ 

The fourth inequality is a consequence of the fact that inner product is preserved under the Fourier transform.

4. (Extra credit) Define the frequency support of a function $f$ to be

$$\text{supp}(\hat{f}) := \left\{ n \in \mathbb{Z} : \hat{f}(n) \neq 0 \right\},$$

where $\hat{f}(n)$ denotes the $n$-th Fourier coefficient. Let $\mathcal{F}$ denote the class of all continuous functions whose frequency support is contained in $[-10, 10]$. Given any “gap” sequence $\{d_k : k \geq 1\} \subseteq \mathbb{N}$, find a continuous function $g$ with the following frequency-replicating feature: for every $f \in \mathcal{F}$,

$$\text{supp}[(\hat{f} g)] = \bigcup_{k=1}^{\infty} A_k,$$

with

$$A_k := \{ a_k + n : n \in \text{supp}(\hat{f}) \}$$

for some integer $a_k$, and

$$\text{dist}(A_k, A_{k'}) \geq d_k + \cdots + d_{k'-1} \text{ for all } k < k'.$$
Solution. For a sequence \( \{a_k\} \) specified by
\[
a_1 = 0, \quad a_2 = 20 + d_1, \quad a_3 = 40 + d_1 + d_2, \ldots, \\
a_k = 20 + a_{k-1} + d_{k-1} = 20(k - 1) + d_1 + \cdots + d_{k-1},
\]
set
\[
g(t) = \sum_{k=1}^{\infty} \frac{e^{i\omega_k t}}{k^2}.
\]
By the Weierstrass \( M \)-test, \( g \) is a continuous function. By construction, \( \text{supp}(\hat{g}) = \{a_k : k \geq 1\} \). We will now show that \( g \) has the required properties.

Since every \( f \in \mathcal{F} \) is a trigonometric polynomial, it matches its Fourier series:
\[
f(x) = \sum_{m \in \mathbb{Z} \cap [-10, 10]} \hat{f}(m)e^{imx}.
\]
Substituting this into the integral expression for \( \hat{fg} \) we find that
\[
\hat{fg}(n) = 2\pi \sum_{m \in \mathbb{Z}} \hat{f}(m)\overline{\hat{g}(n-m)}.
\]
For this last expression to be nonzero, there must exist \( m \in \text{supp}(\hat{f}) \) such that \( n - m \in \text{supp}(\hat{g}) = \{a_k : k \geq 1\} \). This means that \( n = (n - m) + m \in a_k + \text{supp}(\hat{f}) = A_k \) for some \( k \), as desired. Finally we verify that for \( k < k' \),
\[
dist(A_k, A_{k'}) \geq \dist(a_k + [-10, 10], a_{k'} + [-10, 10]) \\
\geq (a_{k'} - 10) - (a_k + 10) \\
= a_{k'} - a_k - 20 = 20(k' - k - 1) + d_k + \cdots + d_{k'-1} \\
\geq d_k + \cdots + d_{k'-1}.
\]
\( \square \)