Math 421/510 Quiz 3 Solution

1. Recall that a linear functional \( \ell \) on a normed vector space \( X \) is \textit{bounded} if there exists a finite constant \( C > 0 \) such that
\[
|\ell(x)| \leq C||x||_X, \quad \text{for all } x \in X.
\]
For a normed vector space \( X \) of your choice, find a linear functional on \( X \) that is bounded and one that is not. Provide adequate reasoning for your answer.

\( \text{(10 points)} \)

\textit{Solution.} Let \( X \) denote the real vector space of all polynomials in \([0, 1]\), equipped with the sup norm. Let us define two linear functionals \( \ell_1 \) and \( \ell_2 \) on \( X \), with
\[
\ell_1(p) = p(1) \quad \text{and} \quad \ell_2(p) = p'(1).
\]
The functional \( \ell_1 \) is bounded, since
\[
|\ell_1(p)| = |p(1)| \leq \sup_{t \in [0,1]} |p'(t)| = ||p||_\infty.
\]
However, \( \ell_2 \) is not. To see this, consider the sequence \( p_n \in X \) given by \( p_n(t) = t^n \). We observe that
\[
||p_n||_\infty = 1 \text{ for all } n \geq 1, \quad \text{whereas} \quad \ell_2(p_n) = p_n'(1) = n.
\]
Since
\[
\frac{|\ell_2(p_n)|}{||p_n||_\infty} = n \to \infty,
\]
\( \ell_2 \) is unbounded. \( \Box \)