1. Given any point $t_0 \in [0, 2\pi]$, show using the uniform boundedness principle that there exists a continuous $2\pi$-periodic function whose Fourier series diverges at $t_0$. We sketched a proof of this result in class. Fill in the details.

**Proof.** Consider the $N$-th Dirichlet kernel

$$D_N(t) := \sum_{n=-N}^{N} e^{int}.$$  

It is a fact that the partial sum sequence $S_N f(t) = (D_N * f)(t)$, where the integral defining the convolution is normalized by a factor $1/2\pi$:

$$S_N f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(t-s)f(s)ds.$$  

We show this in several steps, using contradiction. Suppose for all $f \in C[-\pi, \pi]$, we have $S_N f(t_0) \to f(t_0)$. Then:

(a) The mapping $l_N : f \mapsto S_N f(t_0)$ is linear and bounded from $C[-\pi, \pi] := (C[-\pi, \pi], \|\cdot\|_\infty)$ to $\mathbb{C}$, with a $\sup_N \|l_N\| \leq C < \infty$. This is a result of the uniform boundedness principle.

(b) We show that this implies that $S_N$ is bounded from $C[-\pi, \pi]$ to $C[-\pi, \pi]$, with the bound independent of $N$. Indeed, given $f \in C[-\pi, \pi]$, suppose $|S_N f|$ attains its maximum at $t_1$. Consider the translated function $g(t) := f(t+t_1 - t_0)$, which has $\|g\|_\infty = \|f\|_\infty$ and $S_N g(t_0) = S_N f(t_1)$. Hence

$$\|S_N f\|_\infty = |S_N f(t_1)| = |S_N g(t_0)| \leq C \|g\|_\infty = C \|f\|_\infty.$$  

(c) We state a special case of the Young’s convolution theorem:

**Theorem 1.** Let $(X, \mu)$ be a measure space, and $g$ be a measurable function. The convolution operator $T : f \mapsto f * g$ is bounded from $L^\infty$ to $L^\infty$ if and only if $g \in L^1$. Moreover, $\|T\|_{L^\infty \to L^\infty} = \|g\|_{L^1}$.

Now in our situation, $\sup_N \|S_N\| < \infty$ implies that $\sup_N \|D_N\|_1 < \infty$.  

1
Lastly, we show the above cannot happen. Direct computation shows that

\[ D_N(x) = \frac{\sin\left(N + \frac{1}{2}\right)x}{\sin\left(\frac{1}{2}x\right)}. \]

By considering the integral over \(|x| \in [k\pi/(N + \frac{1}{2}), (k + 1)\pi/(N + \frac{1}{2})]\) for each \(k\), we see \(\|D_N\|_1\) is bounded below by a constant times the first \(N\) terms of the harmonic series. Letting \(N \to \infty\), we have \(\|D_N\|_1 \to \infty\), contradiction to the conclusion above.

\[
2. \text{In class, we introduced the concept of a locally convex space, whose topology is generated by a family of seminorms. When is such a topology equivalent to a metric topology? A norm topology?}\n
Note: If \(X\) is locally convex, it separates points by definition taught in class.

(a) We claim such a topology is a metric topology if and only if it is generated by a countable family of seminorms.

\[
\text{Proof.} \quad (\Leftarrow \Rightarrow) \quad \text{Let } \{p_i\}_{i=1}^{\infty} \text{ be the countable family of seminorms that generates a topology on } X. \text{ Then we define a metric by}
\]

\[ d(x, y) := \sum_{i=1}^{\infty} 2^{-i} \frac{p_i(x - y)}{1 + p_i(x - y)}. \]

It is direct to check that \(d\) is a metric. Indeed, \(d(x, y) \geq 0\), and if \(d(x, y) = 0\), then \(p_i(x - y) = 0\) for all \(i\). Since \(\{p_i\}\) separates points, we have \(x = y\). Symmetry is trivial. For the triangle inequality, refer to the following question: https://math.stackexchange.com/questions/309198/if-d-is-a-metric-then-d-1d-is-also-a-metric.

It remains to show \(d\) generates the same topology as \(\{p_i\}_{i=1}^{\infty}\) does. By translation invariance, it suffice to consider their neighbourhood bases at 0:

\[ B_d(\varepsilon) := \{x \in X : d(x, 0) < \varepsilon\}, \]

\[ \bigcap_{i=1}^{n} B_i(\varepsilon_i) := \{x \in X : p_i(x) < \varepsilon_i \quad \forall 1 \leq i \leq n\}. \]

- Given \(\varepsilon > 0\), take \(N\) such that \(\sum_{i=N+1}^{\infty} 2^{-i} < \varepsilon/2\). Take \(\varepsilon_i := \varepsilon/2\) for all \(1 \leq i \leq N\). Thus if \(p_i(x) < \varepsilon/2\) for all \(1 \leq i \leq N\), we have

\[ d(x, 0) \leq \sum_{i=1}^{N} 2^{-i} \frac{\varepsilon}{2} + \sum_{i=N+1}^{\infty} 2^{-i} \frac{\varepsilon}{2} + \sum_{i=N+1}^{\infty} 2^{-i} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

This implies that

\[ \bigcap_{i=1}^{N} B_i(\varepsilon_i) \subseteq B_d(\varepsilon). \]
• On the other hand, given $\varepsilon_i > 0$ for $i = 1, 2, \ldots, n$, then $d(x, 0) < \varepsilon := \min\{\varepsilon_i : 1 = 1, 2, \ldots, n\}$ implies that $p_i(x) < \varepsilon_i$. Hence

$$B_d(\varepsilon) \subseteq \bigcap_{i=1}^{n} B_i(\varepsilon_i).$$

Therefore they generate the same topology.

(“$\implies$”) Note that $\{B_d(1/n)\}_{n=1}^{\infty}$ forms a neighbourhood base at 0. For each $n$, there is $B_{i,n}(\varepsilon_{i,n}), i = 1, 2, \ldots, K_n$ such that

$$B_d\left(0, \frac{1}{n}\right) \supseteq \bigcap_{i=1}^{K_n} B_{i,n}(\varepsilon_{i,n}).$$

Relabel the countable collection $\{p_{i,n} : 1 \leq i \leq K_n, n \in \mathbb{N}\}$ as $\{p_j\}_{j=1}^{\infty}$. Then $\{p_j\}_{j=1}^{\infty}$ generates the metric topology $d$.

(b) We claim such a topology is a norm topology if and only if it is generated by a finite collection of seminorms.

Proof. (“$\iff$”) Let $\mathcal{P} := \{p_i\}_{i=1}^{N}$ be the finite collection of seminorms that generates a topology on $X$. Then we define a norm by

$$\|x\| := \max\{p_i(x), i = 1, 2, \ldots, N\}.$$

It is direct to check that $\|\cdot\|$ is a norm. Indeed, $\|x\| \geq 0$, and if $\|x\| = 0$, then $p_i(x) = 0$ for all $i$. Since $\{p_i\}$ separates points, we have $x = 0$. Homogeneity and the triangle inequality follows from the corresponding properties of the seminorms.

It remains to show $\|\cdot\|$ generates the same topology as $\{p_i\}_{i=1}^{N}$ does. But this is similar and easier than the countable case.

(“$\implies$”) This is trivial.

3. Let $(X, \Omega, \mu)$ be a $\sigma$-finite measure space, $1 \leq p < \infty$. Suppose that $K : X \times X \to \mathbb{F}$ is an $\Omega \times \Omega$-measurable function such that for $f \in L^p(\mu)$ and almost every $x \in X$, the function $K(x, \cdot)f(\cdot) \in L^1(\mu)$ and

$$Kf(x) = \int K(x, y)f(y)d\mu(y)$$

defines an element $Kf \in L^p(\mu)$. Show that $K$ is a bounded operator on $L^p(\mu)$.

Proof. We first prove a lemma:

Lemma 1. Let $(X, \Omega, \mu)$ be a measure space, and $f$ be a measurable function. Suppose $\int_X f g$ converges absolutely for every $f \in L^p, 1 \leq p \leq \infty$. Then $g \in L^{p'}$, where $p'$ is the dual exponent of $p$. 
4. (a) Show that the weak topology on $X$ is the weakest topology for which all $l \in X^*$ is continuous.

Proof. We take the definition of weak topology on $X$ as the topology generated by the seminorms $p(x) := |l(x)|$ over $l \in X^*$.

Recall that a linear functional $l : X \to \mathbb{F}$ is continuous if and only if there exists finitely many seminorms $p_i$, $1 \leq i \leq n$, and a constant $C$ such that for all $x \in X$,

$$|l(x)| \leq C \sum_{i=1}^{n} p_i(x).$$

Now we take $C = 1$ and a single $p_i = |l|$ to finish the proof.

On the other hand, given any topology on $X$ such that each $l \in X^*$ is continuous. Since taking modulus on the scalar field is continuous, we see that each $x \mapsto p(x) = |l(x)|$ is continuous. Hence the weak topology is weaker than any topology such that each $l \in X^*$ is continuous. Lastly, by taking intersection of all such topologies, we see that the weak topology on $X$ is unique, so it is indeed the weakest topology such that each $l \in X^*$ is continuous. \hfill $\square$

(b) Show that the weak-star topology is the smallest topology on $X^*$ such that for each $x \in X$, the map $l \mapsto l(x)$ is continuous.

Proof. We take the definition of weak-star topology on $X^*$ as the topology generated by the seminorms $q_x(l) := |l(x)|$ over $x \in X$.

Recall that a linear functional $q : X^* \to \mathbb{F}$ is continuous if and only if there exists finitely many seminorms $q_{x_i}$, $1 \leq i \leq n$, and a constant $C$ such that for all $l \in X^*$,

$$|q(l)| \leq C \sum_{i=1}^{n} q_{x_i}(l) = C \sum_{i=1}^{n} |l(x_i)|.$$
Now for each \( x \in X \), \( q(l) = q_x(l) \) is the mapping \( l \mapsto |l(x)| \). We take \( C = 1 \) and a single \( x_1 = x \) to finish the proof.

On the other hand, given any topology on \( X^* \) such that for each \( x \in X \), \( l \mapsto l(x) \) is continuous. Since taking modulus on the scalar field is continuous, we see that each \( l \mapsto q_x(l) = |l(x)| \) is continuous. Hence the weak star topology is weaker than any topology with the aforesaid property. Uniqueness is similar as the above. \( \square \)

5. (a) If \( \mathbb{H} \) is a Hilbert space and \( \{h_n\} \subseteq \mathbb{H} \) is a sequence such that \( h_n \to h \) weakly and \( \|h_n\| \to \|h\| \), then show that \( h_n \to h \) strongly.

**Proof.** Since \( \mathbb{H} \) is self dual, \( h_n \to h \) weakly if and only if for all \( g \in \mathbb{H} \), we have \( \langle h_n, g \rangle \to \langle h, g \rangle \). Taking \( g = h \), we have \( \langle h_n, h \rangle \to \langle h, h \rangle \).

By assumption, \( \langle h_n, h_n \rangle = \|h_n\|^2 \to \|h\|^2 = \langle h, h \rangle \). Therefore

\[
\langle h_n - h, h_n - h \rangle = \langle h_n, h_n \rangle - \langle h_n, h \rangle - \langle h, h_n \rangle + \langle h, h \rangle \\
\to \langle h, h \rangle - \langle h, h \rangle - \langle h, h \rangle + \langle h, h \rangle \\
= 0.
\]

\( \square \)

(b) Prove the same statement for the Lebesgue spaces \( L^p(\mu) \), \( 1 < p < \infty \).

**Proof.** We will use the fact that \( L^p(\mu) \) is uniformly convex for \( 1 < p < \infty \), that is, for each \( 0 < \varepsilon < 1 \), there is \( \delta > 0 \) such that for all \( \|f\|_p = 1 = \|g\|_p \), \( \|f - g\|_p > \varepsilon \) implies that \( \|(f + g)/2\|_p < 1 - \delta \). This is a direct result of the Clarkson’s inequalities (an elementary calculation with \( \varepsilon - \delta \) involved):

\[
\left\| \frac{f + g}{2} \right\|_p^p + \left\| \frac{f - g}{2} \right\|_p^p \leq \frac{1}{2} (\|f\|_p^p + \|g\|_p^p), \quad \text{if} \quad 2 \leq p < \infty; \quad (1)
\]

\[
\left\| \frac{f + g}{2} \right\|_p' + \left\| \frac{f - g}{2} \right\|_p' \leq \left( \frac{1}{2} \|f\|_p^p + \frac{1}{2} \|g\|_p^p \right)^{\frac{1}{p'}}, \quad \text{if} \quad 1 < p < 2; \quad (2)
\]

where \( 1/p + 1/p' = 1 \).


We still need another tool, namely, Fatou’s lemma on weakly convergent sequences:

**Lemma 2.** Let \( x_n \to x \) in a normed space \( X \). Then \( \|x\| \leq \liminf_{n \to \infty} \|x_n\| \).

**Proof of the Lemma.** Using duality, we have \( \|x\| = \sup_{\|f\|_{X^*} = 1} |f(x)| \). Now let \( f \in X^* \) with \( \|f\|_{X^*} = 1 \). We have

\[
|f(x)| = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} |f(x_n)| \leq \liminf_{n \to \infty} \|f\|_{X^*} \|x_n\| = \liminf_{n \to \infty} \|x_n\|.
\]

Since \( f \in X^*, \|f\|_{X^*} = 1 \) is arbitrary, we have \( \|x\| \leq \liminf_{n \to \infty} \|x_n\| \). \( \square \)
We now come to the proof of the analogous statement as above. Since $f_n \rightharpoonup f$, $(f_n + f)/2 \rightharpoonup f$. By Fatou’s lemma on weakly convergent sequences, we have

$$\|f\|_p \leq \liminf_{n \to \infty} \left\| \frac{f_n + f}{2} \right\|_p.$$ 

On the other hand, we also have

$$\left\| \frac{f_n + f}{2} \right\|_p \leq \frac{1}{2} \|f_n\|_p + \frac{1}{2} \|f\|_p \to \|f\|_p,$$

which follows from the assumption that $\|f_n\|_p \to \|f\|_p$. This forces that all the above inequalities should be equalities, whence we have

$$\lim_{n \to \infty} \left\| \frac{f_n + f}{2} \right\|_p = \|f\|_p.$$ 

Lastly, either using the uniform convexity, or just plugging $g = f_n$ in the Clarkson’s inequality which is simpler in this case, and taking limits $n \to \infty$, we have $\|f - f_n\|_p \to 0$. 

6. Suppose that $X$ is an infinite-dimensional normed space. Find the weak closure of the unit sphere.

**Proof.** (Credit to Jeffrey Dawson for this solution)

We claim that the weak closure of the unit sphere $S$ is the closed unit ball $B := \{x \in X : \|x\| \leq 1\}$. (Remark: for a normed space, the closed unit ball is equal to the closure of the (open) unit ball, which is not true for a general metric space.)

We claim that

$$B = \bigcap_{\|l\| = 1} \{x : |l(x)| \leq 1\}.$$ 

Indeed, if $\|x\| \leq 1$, then $|l(x)| \leq 1$ whenever $\|l\| = 1$; on the other hand, if $\|x\| > 1$, then by the Hahn-Banach theorem, there is $l \in X^*$ such that $\|l\| = 1$ and $l(x) = \|x\| > 1$. This proves the claim above.

Since each $\{x : |l(x)| \leq 1\}$ is weakly closed, so is any intersection over $\|l\| = 1$. Hence $B$ is a weakly closed set containing $S$, so $B$ contains the weak closure of $S$.

On the other hand, let $x_0 \in B$; we want to show that $x_0$ is in the weak closure of $S$. To do this, let $G$ be a weakly open set containing $x_0$, and without loss of generality, assume $G$ is a basic weakly open neighbourhood of $x_0$, that is, there are $l_i \in X^*$, $\delta_i > 0$, $1 \leq i \leq n$, such that

$$G = \bigcap_{i=1}^n \{x : |l_i(x - x_0)| < \delta_i\}.$$ 

Now we take $0 \neq y \in \cap_{i=1}^n \text{Ker}(l_i)$; this is possible since the right hand side has codimension $n < \infty$ while $X$ is infinite-dimensional. The functions $\lambda \mapsto \|\lambda y + x_0\|$ is a continuous function which sends $0$ to $\|x_0\| \leq 1$ and tends to $\infty$ as $\lambda \to \infty$. 

6
By the intermediate value theorem, there is $\lambda \geq 0$ such that $\|\lambda y + x_0\| = 1$. Let $x = \lambda y + x_0$, then $\|x\| = 1$ and $l_i(x - x_0) = l_i(\lambda y) = 0$ for all $i$, so $x \in G$, and thus $G \cap S \neq \emptyset$. Since $G$ is arbitrary, $x_0$ is in the weak closure of $S$.

Combining two sides finishes the proof. \qed