1. Let $Y = L^1(\mu)$ where $\mu$ is the counting measure on $\mathbb{N}$, and let $X = \{f \in Y : \sum_{n=1}^{\infty} n|f(n)| < \infty\}$, equipped with $L^1$-norm.

(a) $X$ is a proper dense subspace of $Y$; hence $X$ is not complete.

Proof. 
- It is direct to check that $X$ is a subspace of $Y$.
- $X \nsubseteq Y$, since $f(n) := n^{-2} \in Y$ but not in $X$.
- $X$ is dense in $Y$. Too see this, let $x \in Y$ and $\varepsilon > 0$. Then there is $N$ such that $\sum_{n=N}^{\infty} |f(n)| < \varepsilon$. But the truncated sequence $g(n) := f(n)1_{(n<N)}$ clearly lies in $X$ and satisfies $\sum_{n=1}^{\infty} |f(n) - g(n)| < \varepsilon$.

(b) Define $T : X \to Y$ by $Tf(n) = nf(n)$. Then $T$ is closed but not bounded.

Proof. 
- By definition, $T$ is a closed linear operator (not a closed map!!), if $f_m \to f$ in $X$ and $Tf_m \to g$ in $Y$ implies that $g = Tf$. In our case, we are to show

$$g(n) = nf(n) \quad \forall n \in \mathbb{N},$$

given that

$$\lim_{m \to \infty} \sum_{n=1}^{\infty} |f_m(n) - f(n)| = 0, \quad (1)$$

$$\lim_{m \to \infty} \sum_{n=1}^{\infty} |nf_m(n) - g(n)| = 0, \quad (2)$$

In particular, for any $n \in \mathbb{N}$, (1) implies that $\lim_{m \to \infty} f_m(n) = f(n)$, and (2) implies that $\lim_{m \to \infty} nf_m(n) = g(n)$. Combining these two gives $g(n) = nf(n)$, as desired.

Comment. Many of you proved the statement that $T$ is a topologically closed map. It is an exercise to show that this is stronger than $T$ being a closed linear operator.
Consider $f_m(n) := e_m$ for $m \in \mathbb{N}$, where \( \{e_m\}_{m=1}^\infty \) is the canonical basis for $L^1(\mu)$. Then $\|Tf_m\|_1 = m$, so

$$
\sup_{f \in X, \|f\|_1 = 1} \frac{\|Tf_m\|_1}{\|f_m\|_1} \geq \frac{m}{1} = m.
$$

Since $m$ can arbitrarily large, $T$ is unbounded.

(c) Let $S = T^{-1}$. Then $S : Y \to X$ is bounded and surjective but not open.

Proof.  
- Clearly, $S$ is well defined by $Sf(n) = f(n)/n$. It is bounded since

$$
\|Sf\|_1 = \sum_{n=1}^{\infty} \frac{|f(n)|}{n} \leq \sum_{n=1}^{\infty} |f(n)| = \|f\|_1.
$$

- $S$ is surjective, since given any $f \in X$, we have $Tf \in Y$ and $S(Tf) = f$ by definition.
- $S$ is open if and only if $S^{-1} = T$ is continuous if and only if $T$ is bounded since $T$ is linear. But $T$ is unbounded, so $S$ is not open.

2. Let $Y = C[0,1]$ and $X = C^1[0,1]$, both equipped with the uniform norm.

(a) $X$ is not complete.

Proof. By the Weierstrass approximation theorem, the space of all polynomials $P$ is dense in $Y$ under the sup-norm. Since $P \subseteq X$, that means $X$ is also dense in $Y$. If $X$ is complete, then $X = Y$, which is absurd. Thus $X$ cannot be complete.

(b) The map $(d/dx) : X \to Y$ is closed but not bounded.

Proof.  
- To show the map is closed, let $f_n \to f$ in $X$, $f'_n \to g$ in $Y$, and our goal is to show that $g = f'$. This is proved in Problem 3(b) of Homework 1.
- The map is not bounded, as can be seen from the examples $x^n \mapsto nx^{n-1}$, $n \in \mathbb{N}$.

3. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the vector space $X$ such that $\|\cdot\|_1 \leq \|\cdot\|_2$. If $X$ is complete with respect to both norms, then the norms are equivalent.

Proof. Define $I : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ to be the identity map. This maps is clearly linear and surjective, and $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are both complete by assumption. Moreover, $\|I\|_{op} \leq 1$. By the open mapping theorem, $I$ is open, which means that $I^{-1} : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is continuous, and hence bounded. Thus there is $C$ with $\|\cdot\|_2 \leq C\|\cdot\|_1$, so the norms are equivalent.
4. There is no slowest rate of decay of the terms of an absolutely convergence series; that is, there is no sequence \( \{a_n\} \) of positive numbers such that \( \sum a_n |c_n| < \infty \) if and only if \( \{c_n\} \) is bounded.

**Proof.** Suppose there is such sequence \( \{a_n\} \). Define \( T : B(\mathbb{N}) \rightarrow L^1(\mu) \) by \( Tf(n) = a_n f(n) \), where \( B(\mathbb{N}) \) is the space of all bounded sequences endowed with the sup-norm. The assumption is to say that \( T \) is well defined and invertible, with \( T^{-1} f(n) = a_n^{-1} f(n) \).

The mapping \( T \) is bounded, which we now show. By definition of \( \{a_n\} \), if we take \( c_n = e := (1, 1, 1, \ldots) \in B(\mathbb{N}) \), then we get \( \sum a_n < \infty \). Thus

\[
\|Tf\|_1 = \sum_{n=1}^{\infty} a_n |f(n)| \leq \|f\|_\infty \sum_{n=1}^{\infty} a_n,
\]

so \( T \) is bounded. By the open mapping theorem, \( T \) is open. Therefore \( T \) is a homeomorphism between the spaces \( B(\mathbb{N}) \) and \( L^1(\mu) \).

Consider \( S \), the set of \( f \) such that \( f(n) = 0 \) for all but finitely many \( n \). \( S \) is dense in \( L^1 \), which is proved in Q1 (a). But \( S \) is not dense in \( B(\mathbb{N}) \). For, consider \( e \in B(\mathbb{N}) \). If \( h \in S \) is any finite sequence, then \( \|g - h\|_\infty \geq 1 \).

But \( T \) is a homeomorphism between \( B(\mathbb{N}) \) and \( L^1(\mu) \), and \( S \) is dense in \( L^1(\mu) \), so \( T^{-1}(S) \) is dense in \( B(\mathbb{N}) \). But \( T^{-1}(S) \subseteq S \), so \( S \) is dense in \( B(\mathbb{N}) \), which is a contradiction. Therefore, such positive sequence \( \{a_n\} \) does not exist. \( \square \)

5. Let \( X \) and \( Y \) be Banach spaces. If \( T : X \rightarrow Y \) is a linear map such that \( f \circ T \in X^* \) for every \( f \in Y^* \), then \( T \) is bounded.

**Proof.** Since \( X \) and \( Y \) are Banach spaces, to show that \( T \) is bounded, it is equivalent to showing that \( T \) is a closed linear operator.

Let \( x_n \rightarrow x \) in \( X \) and \( Tx_n \rightarrow y \) in \( Y \). To show that \( Tx = y \), we claim that it is equivalent to showing that \( f(Tx) = f(y) \) for all \( f \in Y^* \), which is exactly our assumption. Indeed, by linearity, if \( Tx - y \neq 0 \), then by a corollary of the Hahn-Banach theorem (Q4 of Homework 2), there is \( f \in Y^* \) such that \( f(Tx - y) = 1 \), which is a contradiction. Hence \( Tx = y \) and \( T \) is closed. \( \square \)

6. Let \( X \) and \( Y \) be Banach spaces, and let \( T_n \) be a sequence in \( L(X,Y) \) such that \( \lim_n T_n x \) exists for every \( x \in X \). Let \( Tx = \lim_n T_n x \); then \( T \in L(X,Y) \).

**Proof.** Let \( x \in X \). Since \( Tx = \lim_n T_n x \) exists, in particular, \( \{T_n x\} \) is bounded in \( n \). Since \( X \) is a Banach space, the uniform boundedness principle implies that \( \sup_n \|T_n\|_{op} \leq M < \infty \). Thus

\[
\|Tx\| = \lim_{n \rightarrow \infty} \|T_n x\| \leq \sup_n \|T_n\|_{op} \|x\| \leq M \|x\|.
\]

Since \( T \) is obviously linear, \( T \in L(X,Y) \). \( \square \)
7. Let $X$ and $Y$ be Banach spaces and $\{T_{jk} : j, k \in \mathbb{N}\} \subseteq L(X, Y)$. Suppose that for each $k$ there exists $x \in X$ such that $\sup\{\|T_{jk}x\| : j \in \mathbb{N}\} = \infty$. Then there is an $x$ such that $\sup\{\|T_{jk}x\| : j \in \mathbb{N}\} = \infty$ for all $k$.

**Proof.** We prove it by contradiction. Suppose there is no such $x$. Then for all $x$, there is $k_x$ such that the sequence $\sup\{\|T_{jk}x\| : j \in \mathbb{N}\} < \infty$. Thus we can write

$$X = \bigcup_{k=1}^{\infty} \{ x : \sup_j \|T_{jk}x\| < \infty \} := \bigcup_{k=1}^{\infty} E_k.$$

Denote $E_{k,n} := \{ x : \sup_j \|T_{jk}x\| \leq n \}$, and hence $X = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} E_{k,n}$.

- Each $E_{k,n}$ is closed: given $x_m \subseteq E_{k,n}$ with $x_m \to x$, then for all $j$ we have

$$\|T_{jk}x\| = \lim_m \|T_{jk}x_m\| \leq n,$$

since $T_{jk}$ is continuous and $x_m \in E_{k,n}$. Hence $x \in E_{k,n}$.

- Each $E_{k,n}$ is nowhere dense. To see this, note first it is easy to check that $E_k$ is a subspace of $X$; moreover, $E_k \subseteq X$ by the assumption that there is $x \in X$ such that $\sup\{\|T_{jk}x\| : j \in \mathbb{N}\} = \infty$. Hence $E_k$ is a proper subspace of $X$, so $E_k$ is nowhere dense. As a subset of $E_k$, $E_{k,n}$ is also nowhere dense.

Since $X$ is a Banach space, we have reached a contradiction to the Baire category theorem. Hence our assumption is false, that is, there is an $x$ such that $\sup\{\|T_{jk}x\| : j \in \mathbb{N}\} = \infty$ for all $k$. \qed