1. Let $M$ denote the Hardy-Littlewood maximal function. Show that if $f$ is not identically zero, then $Mf$ is never integrable on $\mathbb{R}^n$.

2. We proved in class that for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the family of averages

$$ (1) \quad \frac{1}{|B(x; r)|} \int_{B(x; r)} f(y) \, dy $$

admits a limit for almost every $x \in \mathbb{R}^n$ as $r \to 0$. Modify that argument to show that, in fact, the complement of the set

$$ (2) \quad L(f) = \{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{1}{|B(x; r)|} \int_{B(x; r)} |f(y) - f(x)| \, dy = 0 \} $$

is of null Lebesgue measure. Hence deduce that for almost every $x$, the limit of (1) as $r \to 0$ is $f(x)$. The set in (2) is called the Lebesgue set of $f$.

3. For each of the criteria specified below, find an example of a Borel set $E \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$ with this property.

   (a) the limiting value of the averages in (1) does not exist with $f = 1_E$ as $r \to 0$.

   (b) Given any number $\alpha \in (0, 1)$ and $f = 1_E$, the limiting value exists and equals $\alpha$.

4. Let $\mu$ be any regular complex measure on $\mathbb{R}^n$. Discuss the limiting behaviour, as $r \to 0$, of the averages $\mu(B(x; r)/|B(x; r)|$, possibly excluding a class of points $x$ of zero Lebesgue measure.

5. Let $\mathcal{S}$ be a family of measurable sets in $\mathbb{R}^n$ with the following property: for each $x \in \mathbb{R}^n$ and $r > 0$, there exists $S_r(x) \in \mathcal{S}$ satisfying

$$ S_r(x) \subseteq B(x; r) \quad \text{and} \quad |B(x; r)| \leq C|S_r(x)|, $$

for some constant $C > 0$ independent of $r$ and $x$. 
(a) Give at least two distinct examples of families of sets $S$ that meets the two requirements described above. Also provide at least two examples of $S$ which satisfies the first condition but does not meet the second.

(b) Show that
\[ \lim_{r \to 0} \frac{1}{|S_r(x)|} \int_{S_r(x)} |f(y) - f(x)| \, dy = 0 \]
for every point $x$ in the Lebesgue set of $f$.

6. The Hardy Littlewood maximal operator $M$ is of fundamental importance in part because it controls many other operators of interest arising in a variety of contexts. We illustrate this in the context of the Dirichlet problem for Laplace’s equation.

(a) Suppose that $g : \mathbb{R}^d \to [0, \infty]$ is radial and nonincreasing. In other words, $g(x) = h(|x|)$ with $h(r_1) \geq h(r_2)$ for $0 \leq r_1 \leq r_2$. Show that $f * g(x) \leq ||g||_1 Mf(x)$ for all $x$ and all non-negative $f$.

(b) Recall the Poisson kernel for the upper half-space $\mathbb{R}^{n+1}_+ = \{(x, t) : x \in \mathbb{R}^n, t > 0\}$:
\[ p_t(x) = c_n t^{-n} (1 + |t^{-1} x|^2)^{-\frac{n+1}{2}}. \]
Verify that for any bounded continuous $f$ or for $f \in L^p$, $p \in [1, \infty]$, the function $u(x, t) = f * p_t(x)$ obeys Laplace’s equation $\Delta u = 0$ on $\mathbb{R}^{n+1}_+$.

(c) Let’s focus now on the boundary behaviour of $u$. Show that $u(x, t) \to f(x)$ as $t \to 0$ uniformly on compact sets if $f$ is a bounded continuous function. Prove convergence in $L^p$ as $t \to 0$ if $f \in L^p(\mathbb{R}^n)$.

(d) What can we say about the pointwise convergence of $u$ to $f$? Show that $u(x, t) \to f(x)$ as $t \to 0$ non-tangentially for almost every $x \in \mathbb{R}^n$. This means that for almost every $x$, and every $r > 0$,
\[ u(y, t) \to f(x) \quad \text{as} \quad (y, t) \to (x, 0), \quad \text{with} \]
\[ (y, t) \in \Gamma_r(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |x - y| < rt\}. \]