Math 121 Assignment 8
Due Friday April 1

1. Find the centre, radius and interval of convergence of each of the following power series.

(a) \( \sum_{n=0}^{\infty} \frac{1 + 5^n}{n!} x^n \)  
(b) \( \sum_{n=1}^{\infty} \frac{(4x - 1)^n}{n^n} \).

2. Expand
(a) \( \frac{1}{x^2} \) in powers of \( x + 2 \).
(b) \( x^3/(1 - 2x^2) \) in powers of \( x \).
(c) \( e^{2x+3} \) in powers of \( x + 1 \).
(d) \( \sin x - \cos x \) about \( \frac{\pi}{4} \).
(e) the Maclaurin series of \( \ln(e + x^2) \).
(f) the Maclaurin series of \( \cos^{-1} x \).

For each expansion above, determine the interval on which the representation is valid.

3. Find the sums of the following series.

(a) \( \sum_{n=0}^{\infty} \frac{(n + 1)^2}{\pi^n} \)  
(b) \( \sum_{n=1}^{\infty} \frac{(-1)^n n(n + 1)}{2^n} \)  
(c) \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n} \).
(d) \( \frac{x^3}{3!} \times 4 + \frac{x^5}{5!} \times 16 - \frac{x^7}{7!} \times 64 + \frac{x^9}{9!} \times 256 - \cdots \)
(e) \( 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \frac{x^8}{9!} + \cdots \)
(f) \( 1 + \frac{1}{2 \times 2!} + \frac{1}{4 \times 3!} + \frac{1}{8 \times 4!} + \cdots \)
(g) \( 1 - \frac{x}{2!} + \frac{x^2}{4!} - \cdots \)

4. This problem outlines a strategy for verifying whether a function \( f \) is real-analytic. Recall the \( n \)th order Taylor polynomial of \( f \) centred at \( c \):

\[
P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x - c)^k,
\]
and set $E_n = f(x) - P_n(x)$.

(a) Use mathematical induction to show that

$$E_n(x) = \frac{1}{n!} \int_c^x (x-t)^n f^{(n+1)}(t) \, dt,$$

provided $f^{(n+1)}$ exists on an interval containing $c$ and $x$. The formula above is known as Taylor’s formula with integral remainder.

(b) Use Taylor’s formula with integral remainder to prove that $\ln(1+x)$ is real analytic at $x = 0$; more precisely, that the Maclaurin series of $\ln(1+x)$ converges to $\ln(1+x)$ for $-1 < x \leq 1$.

5. Find the Maclaurin series for the functions:

(a) $L(x) = \int_1^{1+x} \frac{\ln t}{t-1} \, dt$

(b) $M(x) = \int_0^x \frac{\tan^{-1} t^2}{t^2} \, dt$

6. Evaluate the limits

(a) $\lim_{x \to 0} \frac{(e^x - 1 - x)^2}{x^2 - \ln(1 + x^2)}$

(b) $\lim_{x \to 0} \frac{\sin(\sin x) - x}{x(\cos(\sin x) - 1)}$

(c) $\lim_{x \to 0} \frac{x^3 - 3S(x)}{x^7}$ where $S(x) = \int_0^x \sin(t^2) \, dt$.

(d) $\lim_{x \to 0} \frac{(x - \tan^{-1} x)(e^{2x} - 1)}{2x^2 - 1 + \cos(2x)}$.

7. (a) Estimate the size of the error if the Taylor polynomial of degree 4 about $x = \pi/2$ for $f(x) = \ln \sin x$ is used to approximate $\ln \sin(1.5)$. 
(b) How many nonzero terms of the Maclaurin expansion of $e^{-x^4}$ are needed to evaluate $\int_{0}^{1/2} e^{-x^4} \, dx$ correct to five decimal places? Evaluate the integral to that accuracy.

8. Find the Fourier series of the 3-periodic function

$$f(x) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ 1 & \text{if } 1 \leq t < 2 \\ 3 - t & \text{if } 2 \leq t < 3. \end{cases}$$

9. Verify that if $f$ is an even function of period $T$, then the Fourier sine coefficients $b_n$ of $f$ are all zero and the Fourier cosine coefficients $a_n$ of $f$ are given by

$$a_n = \frac{4}{T} \int_{0}^{T/2} f(t) \cos(n \omega t) \, dt, \quad n = 0, 1, 2, \ldots$$

where $\omega = 2\pi/T$. State and verify the corresponding result for odd functions $f$.

10. Prove that the binomial coefficients satisfy:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$