Math 121 Assignment 6
Due Friday March 4

1. If 100 N.cm of work must be done to compress an elastic spring to 3 cm shorter than its normal length, how much work must be done to compress it 1 cm further? Recall that by Hooke’s law, the force required to compress an elastic spring to \(x\) units shorter than its natural length is proportional to \(x\).

2. A bucket is raised vertically from ground level at a constant speed of 2m/min by a winch. If the bucket weighs 1 kg and contains 15 kg of water when it starts up but loses water by leakage at the rate of 1 kg/min thereafter, how much work must be done by the winch to raise the bucket to a height of 10m?

3. For each of the following equations, find a function \(y(x)\) that obeys it.
   (a) \(y(x) = 3 + \int_0^x e^{-y(t)} \, dt\).
   (b) \(x^2y' + y = x^2e^{1/x}, \ y(1) = 3e\).

4. Find the equation of a curve that passes through the point \((2, 4)\) and has slope \(3y/(x-1)\) at any point on it.

5. The initial balance in the account was $1000. Interest is paid continuously into the account at a rate of 10% per annum, compounded continuously. The account is also being continuously depleted by taxes at the rate of \(y^2/10 - 6\) dollars per year, where \(y = y(t)\) is the balance in the account after \(t\) years. How large can the account grow? How long will it take the account to grow to half its balance?

6. Identify the parametric curves
   \[
   \begin{align*}
   x &= \cosh t \\
   y &= \sinh^2 t
   \end{align*}
   \]
   \[
   \begin{align*}
   x &= \cos t + \sin t \\
   y &= \cos t - \sin t.
   \end{align*}
   \]

7. For the following two examples, determine the points where the given parametric curves have horizontal and vertical tangents.
   \[
   \begin{align*}
   x &= \frac{4}{1+t^2} \\
   y &= t^3 - 3t
   \end{align*}
   \]
   \[
   \begin{align*}
   x &= t^3 - 3t \\
   y &= t^3 - 12t.
   \end{align*}
   \]
8. Find the length of the curve \( x = e^t - t, \ y = 4e^{t/2} \) from \( t = 0 \) to \( t = 2 \).

9. Sketch the polar graph of the equation \( r = 1 + 2 \cos 2\theta \) and find the area of one of the two smaller loops.

10. Find the area of the region inside the cardioid \( r = 1 + \cos \theta \) and to the left of the line \( x = 1/4 \).

11. Show that a plane that is not parallel to the axis of a circular cylinder intersects the cylinder in an ellipse.

12. At what points do the curves \( r^2 = 2 \sin 2\theta \) and \( r = 2 \cos \theta \) intersect? At what angle do the curves intersect at each of these points?

13. A tractrix is a curve in the first quadrant of the \((x, y)\) plane, starting from the point \((L, 0)\), and having the property that if the tangent line to the curve at \( P \) meets the \( y \)-axis at \( Q \), then the length of \( PQ \) is the constant \( L \). (For example, think of a trailer of length \( L \) attached to a tractor which is sitting at the origin. The rear end \( P \) of the trailer was originally lying at \((L, 0)\). As the tractor moves away along the \( y \)-axis, the path traced out by \( P \) is a tractrix.) Find the equation of the tractrix.