Math 121: Homework 8 solutions

1. (a) For \( \sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n \) we have \( R = \lim_{n \to \infty} \frac{(n+1)!}{1+5^{n+1}} = \infty. \) The radius of convergence is infinite, the center of convergence is 0. The interval of convergence is the whole real line \((-\infty, \infty).\)

(b) We have \( \sum_{n=1}^{\infty} \frac{(4x-1)^n}{n!} = \sum_{n=1}^{\infty} \frac{4^n}{n!} (x - 1/4)^n. \) The center of convergence is \( x = 1/4. \) The radius of convergence is \( R = \lim_{n \to \infty} \frac{4^n (n+1)^{n+1}}{n+1} = \infty. \)

Hence, the interval of convergence is \((-\infty, \infty).\)

2. (a) Let \( x + 2 = t, \) so \( x = t - 2. \) Then

\[
\frac{1}{x^2} = \frac{1}{(2-t)^2} = \sum_{n=0}^{\infty} \frac{(n+1)t^n}{2^{n+2}} = \sum_{n=0}^{\infty} \frac{(n+1)(x+2)^n}{2^{n+2}},
\]

\((-4 < x < 0).\)

(b) We have

\[
\frac{x^3}{1-2x^2} = x^3 \left( \sum_{n=0}^{\infty} (2x^2)^n \right) = \sum_{n=0}^{\infty} 2^n x^{2n+3},
\]

\((-1/\sqrt{2} < x < 1/\sqrt{2}).\)

(c) Let \( t = x + 1. \) Then \( x = t - 1, \) and

\[
e^{2x+3} = e^{2t+1} = e^{2t} = e \sum_{n=0}^{\infty} \frac{2^n t^n}{n!} \quad (\text{forall } t)
\]

\[
= \sum_{n=0}^{\infty} \frac{e2^n(x+1)^n}{n!} \quad (\text{forall } x)
\]

(d) Let \( t = x - \pi/4, \) so \( x = t + \pi/4. \) Then

\[
f(x) = \sin x - \cos x
\]

\[
= \sin(t + \pi/4) - \cos(t + \pi/4)
\]

\[
= \frac{1}{\sqrt{2}} \left[ (\sin t + \cos t) - (\cos t - \sin t) \right]
\]

\[
= \sqrt{2} \sin t = \sqrt{2} \sum_{n=0}^{\infty} (-1)^n \frac{2^{n+1}}{(2n+1)!}
\]

\[
= \sqrt{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x - \pi/4)^{2n+1}.
\]

For all \( x. \)
\[
\ln(e + x^2) = \ln e + \ln(1 + \frac{x^2}{e}) = \ln e + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{ne^{2n}}.
\]

\(-\sqrt{e} < x \leq \sqrt{e}\).

(f) \[
\arccos x = \frac{\pi}{2} - \arcsin x = \frac{\pi}{2} - (x + \frac{1}{2} x^3 + \frac{13}{24} x^5 + \ldots) \quad -1 < x < 1.
\]

3. (a) \[
\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1 + x}{(1-x)^3},
\]

for \(-1 < x < 1\). Putting \(x = 1/\pi\), we get \[
\sum_{n=0}^{\infty} \frac{(n+1)^2}{\pi^n} = \sum_{k=1}^{\infty} \frac{k^2}{\pi^{k-1}} = \frac{1 + 1/\pi}{(1 - 1/\pi)^3} = \frac{\pi(\pi+1)}{(\pi-1)^3}.
\]

(b) \[
\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}, \quad -1 < x < 1.
\]

Differentiate with respect to \(x\) and then replace \(n\) by \(n+1\):

\[
\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}, \quad -1 < x < 1.
\]

\[
\sum_{n=1}^{\infty} n(n+1)x^{n-1} = \frac{2}{(1-x)^3}, \quad -1 < x < 1.
\]

Now let \(x = -1/2\):

\[
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n(n+1)}{2^{n-1}} = \frac{16}{27}.
\]

Finally, multiply by \(-1/2\):

\[
\sum_{n=1}^{\infty} (-1)^n \frac{n(n+1)}{2^n} = -\frac{8}{27}.
\]

(c) Since \[
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \ln(1 + x),
\]

for \(-1 < x \leq 1\), therefore

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n} = \ln(1 + 1/2) = \ln(3/2).
\]
(d) \[ \sum = 2 \left[ \frac{x^3}{2} - \frac{1}{3!} \left( \frac{x^3}{2} \right)^3 + \ldots \right] = 2 \sin \left( \frac{x^3}{2} \right) \]

for all \( x \).

(e) \[ \sum = \frac{1}{x} \sinh x = \frac{e^x - e^{-x}}{2x} , \]

if \( x \neq 0 \). The sum is 1 if \( x = 0 \).

(f) \[ \sum = 2 \left[ \frac{1}{2} + \frac{1}{2!} (1/2)^2 + \frac{1}{3!} (1/2)^3 + \ldots \right] = 2 \left( e^{1/2} - 1 \right) . \]

(g) \[ \sum = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} . \]

The series is the Maclaurin series for \( \cos x \) with \( x^2 \) replaced by \( x \). For \( x > 0 \) the series therefore represents \( \cos \sqrt{x} \). For \( x < 0 \), the series is \( \sum_{n=0}^{\infty} \frac{|x|^n}{(2n)!} \), which is the Maclaurin series for \( \cosh \sqrt{|x|} \).

4. The Fundamental Theorem of Calculus written in the form

\[ f(x) = f(c) + \int_{c}^{x} f'(t) dt = P_0(x) + E_0(x) \]

is the case \( n = 0 \) of the above formula. We now apply integration by parts to the integral, setting

\[ U = f'(t), \quad dV = dt \]
\[ dU = f''(t) dt, \quad V = -(x-t). \]

We have

\[ f(x) = f(c) - f'(t)(x-t)\big|_{c}^{x} + \int_{c}^{x} (x-t)f''(t) dt \]
\[ = f(c) + f'(c)(x-c) + \int_{c}^{x} (x-t)f''(t) dt \]
\[ = P_1(x) + E_1(x) . \]

We have now proved the case \( n = 1 \) of the formula. We complete the proof for general \( n \) by mathematical induction. Suppose the formula holds for some \( n = k \):

\[ f(x) = P_k(x) + E_k(x) = P_k(x) + \frac{1}{k!} \int_{c}^{x} (x-t)^k f^{(k+1)}(t) dt. \]
Again we integrate by parts, let

\[ U = f^{(k+1)}(t), \]
\[ dV = (x - t)^k dt, \]
\[ dU = f^{(k+2)}(t) dt, \]
\[ V = -\frac{1}{k+1} (x - t)^{k+1}. \]

We have

\[ f(x) = P_k(x) + \frac{f^{(k+1)}(c)}{(k+1)!} (x - c)^{k+1} + \frac{1}{(k+1)!} \int_c^x (x - t)^{k+1} f^{(k+2)}(t) dt. \]

If \( f(x) = \ln(1 + x) \), then

\[ f'(x) = \frac{1}{1 + x} \]
\[ f''(x) = -\frac{1}{(1 + x)^2} \]
\[ f'''(x) = \frac{2}{(1 + x)^3} \]
\[ f^{(4)}(x) = -\frac{3!}{(1 + x)^4}, \ldots \]
\[ f^{(n)}(x) = \frac{(-1)^{n-1}(n - 1)!}{(1 + x)^n}, \]

and

\[ f(0) = 0 \]
\[ f'(0) = 1 \]
\[ f''(0) = -1 \]
\[ f'''(0) = 2 \]
\[ f^{(4)}(0) = -3! \]
\[ f^{(n)}(0) = (-1)^{n-1}(n - 1)!. \]

Therefore, the Taylor Formula is

\[ f(x) = x + \frac{-1}{2!} x^2 + \frac{2}{3!} x^3 + \frac{-3!}{4!} x^4 + \ldots + \frac{(-1)^{n-1}(n - 1)!}{n!} x^n + E_n(x), \]

\[ E_n(x) = \frac{(-1)^{n+1}}{(n+1)!} (x - c)^{n+1} f^{(n+1)}(c). \]
where

\[ E_n(x) = \frac{1}{n!} \int_0^x (x-t)^n f^{(n+1)}(t)dt \]
\[ = \frac{1}{n!} \int_0^x (x-t)^n \frac{(-1)^n n!}{(1+t)^{n+1}}dt \]
\[ = (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}dt. \]

If \(0 \leq t \leq x \leq 1\), then \(1 + t \geq 1\) and

\[ |E_n(x)| \leq \int_0^x |x-t|^n dt = \frac{x^{n+1}}{n+1} \leq \frac{1}{n+1} \to 0 \]
as \(n \to \infty\).

If \(-1 < x \leq t \leq 0\), then

\[ \left|\frac{x-t}{1+t}\right| = \frac{t-x}{1+t} \leq |x|, \]
because \(\frac{t-x}{1+t}\) increases from 0 to \(-x = |x|\) as \(t\) increases from \(x\) to 0. Thus,

\[ |E_n(x)| < \frac{1}{1+x} \int_0^{|x|} |x|^n dt = \frac{|x|^{n+1}}{1+x} \to 0 \]
as \(n \to \infty\) since \(|x| < 1\). Therefore,

\[ f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \]

for \(-1 < x \leq 1\).

5.

\[ K(x) = \int_1^{1+x} \ln \frac{t}{t-1} dt, \quad u = t-1 \]
\[ = \int_0^x \ln(1+u) du \]
\[ = \int_0^x [1 - \frac{u}{2} + \frac{u^2}{3} - \frac{u^3}{4} + ...] du \]
\[ = x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + ... \]
\[ = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)^2}, \quad -1 \leq x \leq 1 \]
\[ M(x) = \int_0^x \frac{\tan^{-1}(t^2)}{t^2} \, dt \]
\[ = \int_0^x \left[ 1 - \frac{t^4}{3} + \frac{t^8}{5} - \frac{t^{12}}{7} + \ldots \right] \, dt \]
\[ = x - \frac{x^5}{3 \times 5} + \frac{x^9}{5 \times 9} - \frac{x^{13}}{7 \times 13} + \ldots \]
\[ = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n+1)(4n+1)}, \quad -1 \leq x \leq 1. \]

6. (a)

\[
\lim_{x \to 0} \frac{(e^x - 1 - x)^2}{x^2 - \ln(1 + x^2)} = \lim_{x \to 0} \frac{(\frac{x^2}{2} + \frac{x^4}{3} + \ldots)^2}{x^2 - \frac{x^6}{3} + \ldots} = \lim_{x \to 0} \frac{\frac{x^4}{3} (1 + x/3 + x^2/12 + \ldots)^2}{x^4/2 - x^6/3 + x^8/4 - \ldots} = \frac{1}{2}
\]

(b)

\[
\lim_{x \to 0} \frac{\sin x - 1/3! \sin^3 x + 1/5! \sin^5 x - \ldots}{x[1 - 1/2! \sin^2 x + 1/4! \sin^4 x - \ldots - 1]} = \lim_{x \to 0} \frac{-\frac{2}{3} x^3 + \text{higher degree terms}}{-\frac{1}{2!} x^3 + \text{higher degree terms}} = \frac{2}{3}
\]

(c)

\[ S(x) = \int_0^x \sin(t^2) \, dt \]
\[ = \int_0^x (t^2 - t^6/3 + \ldots) \, dt \]
\[ = \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \ldots \]
\[ \lim_{x \to 0} \frac{x^3 - 3S(x)}{x^7} = \frac{x^3 - 3x^3 + x^7}{x^7/14} = \frac{1}{14}. \]
7. If \( f(x) = \ln(\sin x) \), then calculation of successive derivatives leads to

\[
f^{(5)}(x) = 24 \csc^4 x \cot x - 8 \csc^2 x \cot x.
\]

Observe that \( 1.5 < \pi/2 \approx 1.5708 \), that \( \csc x \geq 1 \) and \( \cot x \geq 0 \), and that both functions are decreasing on that interval. Thus

\[
|f^{(5)}(x)| \leq 24 \csc^4(1.5) \cot(1.5) \leq 2
\]

for \( 1.5 \leq x \leq \pi/2 \). Therefore, the error in the approximation

\[
\ln(\sin 1.5) \approx P_4(x),
\]

where \( P_4 \) is the 4th degree Taylor polynomial for \( f(x) \) about \( x = \pi/2 \), satisfies

\[
|error| \leq \frac{2}{5!} |1.5 - \pi/2|^5 \leq 3 \times 10^{-8}.
\]

\[
\int_0^{1/2} e^{-x^4} \, dx = \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n+1}(4n+1)n!}.
\]

The series satisfies the conditions of the alternating series test, so if we truncate after the term for \( n = k - 1 \), then the error will satisfy

\[
|error| \leq \frac{1}{2^{4k+1}(4k+1)k!}.
\]

This is less than 0.000005 if \( 2^{4k+1}(4k+1)k! > 200000 \), which happens if \( k \geq 3 \). Thus, rounded to five decimal places,

\[
\int_0^{1/2} e^{-x^4} \, dx \approx \frac{1}{2 \cdot 1 \cdot 1} - \frac{1}{32 \cdot 5 \cdot 1} + \frac{1}{512 \cdot 9 \cdot 2} \approx 0.49386.
\]
8. If \( f \) is even, so its Fourier sine coefficients are all zero. Its cosine coefficients are

\[
\frac{a_0}{2} = \frac{1}{2} \cdot \frac{2}{3} \int_0^3 f(t) dt = \frac{2}{3}.
\]

\[
a_n = \frac{2}{3} \int_0^3 f(t) \cos \frac{2n\pi t}{3} dt = \frac{3}{2n^2\pi^2} \left[ \cos \left( \frac{2n\pi}{3} \right) - 1 - \cos(2n\pi) + \cos \left( \frac{4n\pi}{3} \right) \right].
\]

The latter expression was obtained using Maple to evaluate the integral. If \( n = 3k \), where \( k \) is an integer, then \( a_n = 0 \). For other integers \( n \) we have \( a_n = -9/(2\pi^2n^2) \).

Thus the Fourier series of \( f \) is

\[
\frac{2}{3} - \frac{9}{2\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} \cos \left( \frac{2n\pi t}{3} \right) + \frac{1}{2\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} \cos(2n\pi t).
\]

9. If \( f \) is even and has period \( T \), then

\[
b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \frac{2n\pi t}{T} dt = \frac{2}{T} \left[ \int_0^{T/2} f(t) \sin \frac{2n\pi t}{T} dt + \int_0^{T/2} f(t) \sin \frac{2n\pi t}{T} dt \right].
\]

In the first integral in the line above replace \( t \) with \(-t\). Since \( f(-t) = f(t) \) and sine is odd, we get

\[
b_n = \frac{2}{T} \left[ \int_0^{T/2} f(t)(-\sin \frac{2n\pi t}{T})(-dt) + \int_0^{T/2} f(t) \sin \frac{2n\pi t}{T} dt \right]
\]

\[
= \frac{2}{T} \left[ -\int_0^{T/2} f(t) \sin \frac{2n\pi t}{T} dt + \int_0^{T/2} f(t) \sin \frac{2n\pi t}{T} dt \right]
\]

\[
= 0.
\]

Similarly, we have

\[
a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos \frac{2n\pi t}{T} dt.
\]

The corresponding result for an odd function \( f \) states that \( a_n = 0 \) and

\[
b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin \frac{2n\pi t}{T} dt,
\]

and is proved similarly.

10.

\[
C_0^n = \frac{n!}{0!n!} = 1,
\]

\[
C_n^n = \frac{n!}{n!0!} = 1.
\]
If $0 \leq k \leq n$, then

\[
\binom{k-1}{0} + \binom{k}{n} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k+1)!} (k + (n - k + 1)) = \frac{(n + 1)!}{k!(n + 1 - k)!} = \binom{k}{n+1}.
\]