Math 121: Homework 5 solutions

1. Let a circular disk with radius $a$ have center at point $(a, 0)$. Then the disk is rotated about the $y$–axis which is one of its tangent lines. The volume is:

$$V = 2 \times 2\pi \int_{0}^{2a} x \sqrt{a^2 - (x-a)^2} \, dx.$$ 

Let $u = x-a, \, du = dx$. Then we have:

$$V = 4\pi \int_{-a}^{a} u \sqrt{a^2 - u^2} \, du$$

$$= 4\pi \int_{-a}^{a} u \sqrt{a^2 - u^2} \, du + 4\pi a \int_{-a}^{a} \sqrt{a^2 - u^2} \, du$$

$$= 0 + 4\pi a \left(\frac{1}{2} \pi a^2\right)$$

$$= 2\pi^2 a^3.$$ 

2. The region is symmetric about $x = y$ so has the same volume of revolution about the two coordinate axes. The volume of revolution about the $y$–axis is

$$V = 2\pi \int_{0}^{8} x(4 - x^{2/3})^{3/2} \, dx.$$ 

Let $x = 8 \sin^3 u, \, dx = 24 \sin^2 u \cos u \, du$. Thus, we have

$$V = 3072\pi \int_{0}^{\pi/2} \sin^5 u \cos^4 u \, du$$

$$= 3072\pi \int_{0}^{\pi/2} (1 - \cos^2 u)^2 \cos^4 u \sin u \, du.$$ 

Let $v = \cos u, \, dv = -\sin u \, du$. So

$$V = 3072\pi \int_{0}^{1} (1 - v^2)^2 v^4 \, dv$$

$$= 3072\pi \int_{0}^{1} (v^4 - 2v^6 + v^8) \, dv$$

$$= 3072\pi \left(\frac{1}{5} - \frac{2}{7} + \frac{1}{9}\right) = \frac{8192\pi}{105}.$$ 

3. The volume between height 0 and height $z$ is $z^3$. Thus,

$$z^3 = \int_{0}^{z} A(t) \, dt,$$

where $A(t)$ is the cross-sectional area at height $t$. Differentiating the above equation with respect to $z$, we get $3z^2 = A(z)$. The cross-sectional area at height $z$ is $3z^2 \text{ sq.units.}$
4. (a)\[V = 2 \int_0^r (2\sqrt{r^2 - y^2})^2 dy\]
\[= 8 \int_0^r (r^2 - y^2) dy\]
\[= 8(r^2y - \frac{y^3}{3})|_0^r\]
\[= \frac{16r^3}{3}\text{cu.units.}\]

(b) The area of an equilateral triangle of base 2\(y\) is \(\frac{1}{2}(2y)(\sqrt{3}y) = \sqrt{3}y^2\). Hence, the solid has volume
\[V = 2 \int_0^r \sqrt{3}(r^2 - x^2) dx\]
\[= 2\sqrt{3}(r^2x - \frac{1}{3}x^3)|_0^r\]
\[= \frac{4}{\sqrt{3}}r^3\text{cu.units.}\]

5. (a) \(y = x^2, 0 \leq x \leq 2, y' = 2x\). So we have
\[L = \int_0^2 \sqrt{1 + 4x^2} dx.\]

Let \(2x = \tan \theta, 2dx = \sec^2 \theta d\theta.\)
\[L = \int \frac{1}{2} \frac{x = 0}{2x = 2} \sec^3 \theta d\theta\]
\[= \frac{1}{4} \left(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|\right)|_{x=0}^{x=2}\]
\[= \frac{1}{4} (2x\sqrt{1 + 4x^2} + \ln(2x + \sqrt{1 + 4x^2}))|_0^2\]
\[= \frac{1}{4} (4\sqrt{17} + \ln(4 + \sqrt{17}))\]
\[= \sqrt{17} + \frac{1}{4} \ln(4 + \sqrt{17})\text{units.}\]

(b) \(y = \ln \frac{e^x - 1}{e^x + 1}, 2 \leq x \leq 4,\)
\[y' = \frac{2e^x}{e^{2x} - 1}.\]
The length of the curve is
\[ L = \int_2^4 \sqrt{1 + \frac{4e^{2x}}{(e^{2x} - 1)^2}} \, dx \]
\[ = \int_2^4 \frac{e^{2x} + 1}{e^{2x} - 1} \, dx \]
\[ = \int_2^4 \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx = \ln |e^x - e^{-x}|_2^4 \]
\[ = \ln(e^4 - \frac{1}{e^4}) - \ln(e^2 - \frac{1}{e^2}) \]
\[ = \ln e^4 + \frac{1}{e^2} \text{ units.} \]

6.
\[ S = 2\pi \int_0^1 |x|\sqrt{1 + \frac{1}{x^2}} \, dx = 2\pi \int_0^1 \sqrt{1 + x^2} \, dx. \]
Let \( x = \tan \theta, \, dx = \sec^2 \theta \, d\theta, \)
\[ S = 2\pi \int_0^{\pi/4} \sec^3 \theta \, d\theta \]
\[ = \pi (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)|_{0}^{\pi/4} \]
\[ = \pi [\sqrt{2} + \ln(\sqrt{2} + 1)] \text{ sq.units.} \]

7. (a) The mass of the plate is
\[ m = 2\int_0^4 ky \sqrt{4 - y} \, dy, \]
let \( u = 4 - y, \, du = -dy. \) Then we have
\[ m = 2k \int_0^4 (4 - u)u^{1/2} \, du \]
\[ = 2k(\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2})|_0^4 = \frac{256k}{15}. \]
By symmetry, \( M_{x=0} = 0, \) so \( \bar{x} = 0. \)
\[ M_{y=0} = 2\int_0^4 ky^2 \sqrt{4 - y} \, dy, \]
let \( u = 4 - y, \, du = -dy. \)
\[ M_{y=0} = 2k \int_0^4 (16u^{1/2} - 8u^{3/2} + u^{5/2}) \, du \]
\[ = 2k(\frac{32}{3}u^{3/2} - \frac{16}{5}u^{5/2} + \frac{2}{7}u^{7/2})|_0^4 \]
\[ = \frac{4096k}{105}. \]
Thus, \( \bar{y} = \frac{16}{7}. \) The center of mass of the plate is \((0, \frac{16}{7}).\)
(b) The mass of the ball is

\[ m = \int_{-R}^{R} (y + 2R) \pi (R^2 - y^2) dy \]
\[ = 4 \pi R \left( R^2 y - \frac{y^3}{3} \right) \bigg|_{-R}^{R} \]
\[ = \frac{8}{3} \pi R^4 kg. \]

By symmetry, the center of mass lies along the y-axis; we need only calculate \( \bar{y} \).

\[ M_{y=0} = \int_{-R}^{R} y (y + 2R) \pi (R^2 - y^2) dy \]
\[ = \frac{2}{3} \pi R^5. \]

Thus, \( \bar{y} = \frac{R}{10} \).

(c) A slice at height \( z \) has volume \( dV = \pi y^2 dz \) and density \( \frac{kzg}{cm^3} \). Thus, the mass of the cone is

\[ m = \int_{0}^{b} k z \pi y^2 dz \]
\[ = \pi k a^2 \int_{0}^{b} z (1 - z/b)^2 dz \]
\[ = \pi k a^2 \left( \frac{z^2}{2} - \frac{2z^3}{3b} + \frac{z^4}{4b^2} \right) \bigg|_{0}^{b} \]
\[ = \frac{1}{12} \pi k a^2 b^2 g. \]

The moment about \( z = 0 \) is

\[ M_{z=0} = \pi k a^2 \int_{0}^{b} z^2 (1 - z/b)^2 dz = \frac{1}{30} \pi k a^2 b^3 g - cm. \]

Thus, \( z = \frac{2b}{5} \). Hence, the center of mass is on the axis of the cone at height \( 2b/5cm \) above the base.