1. Recall Jordan’s theorem: a function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if $f$ can be written as the difference of two non-decreasing functions $g$ and $h$.
   (a) Show that the decomposition $f = g - h$ is by no means unique, and that there are uncountably many ways of writing $f$ in this form.
   
   (b) The following decomposition of $f$ is often useful. Define the positive and negative variations of $f$ by
   
   $$ p(x) = \frac{1}{2}(v(x) + f(x) - f(a)), \quad n(x) = \frac{1}{2}(v(x) - f(x) + f(a)), $$
   
   where $v(x) = V^x_a f$ is the variation function defined in class. Show that $p$ and $n$ are nondecreasing functions on $[a, b]$ and use this to give an alternative representation of $f$ as the difference of nondecreasing functions.
   
   (c) The relevance of $p$ and $n$ is that it injects a certain amount of uniqueness into the Jordan decomposition of $f$, in the following sense. If $g$ and $h$ are any two non-decreasing functions on $[a, b]$ such that $f = g - h$, then
   
   $$ V^y_x p \leq V^y_x g \quad \text{and} \quad V^y_x n \leq V^y_x h \text{ for all } x < y \text{ in } [a, b]. $$
   
   Prove this.

2. We stated the “integration by parts” formula in class, using it to highlight the interchanga-
   bility of integrand and integrator. The purpose of this problem is to fill in the details of its
   proof. Throughout this problem $f$ and $\alpha$ denote arbitrary real-valued functions on $[a, b]$.
   
   (a) Given any partition $P = \{ a = x_0 < x_1 < x_2 < \cdots < x_n = b \}$ and a collection of points
   
   $T = \{ t_1, \cdots, t_n \}$ with $t_j \in [x_{j-1}, x_j]$, prove the following identity:
   
   $$ S_f(\alpha, P, T) = f(b)\alpha(b) - f(a)\alpha(a) - S_\alpha(f, P', T'). $$
   
   Here $P' = \{ a = t_0, t_1, \cdots, t_n, t_{n+1} = b \}$ and $T' = P$.
   
   (b) Use part (a) to show that $f \in \mathcal{R}_\alpha[a, b]$ if and only if $\alpha \in \mathcal{R}_f[a, b]$. Show that in either case,
   
   $$ \int_a^b f \, d\alpha + \int_a^b \alpha \, df = f(b)\alpha(b) - f(a)\alpha(a). $$
   
   Note that this is one of those rare instances where one implication implies the other!

3. Let $\alpha \in \text{BV}[a, b]$ and let $\beta(x) = V^x_a \alpha$. Recall that both $\beta$ and $\beta - \alpha$ are increasing. Show that $\mathcal{R}_\alpha[a, b] = \mathcal{R}_\beta[a, b] \cap \mathcal{R}_{\beta - \alpha}[a, b]$. This identity was instrumental to our conclusion that $\mathcal{R}_\alpha[a, b]$ is a vector space, an algebra and a lattice. (Hint: Argue that it suffices to only show that $\mathcal{R}_\alpha[a, b] \subseteq \mathcal{R}_\beta[a, b]$.)

4. Let $\{ f_n \}$ be a bounded sequence in $\text{BV}[a, b]$, i.e., suppose that $\|f_n\|_{\text{BV}} \leq K$ for all $n$.
   Show that $f_n$ admits a pointwise convergent subsequence whose limit $f$ lies in $\text{BV}[a, b]$ with $\|f\|_{\text{BV}} \leq K$. This is known as Helly’s first theorem. (Hint: First try out the case when all the functions $f_n$ are non-decreasing, then adapt it for functions of bounded variation.)