1. Weierstrass’s second theorem states that any continuous $2\pi$-periodic function $f$ on $\mathbb{R}$ is uniformly approximable by trigonometric polynomials. The aim of this exercise is to prove this statement.

(a) Deduce Weierstrass’s second theorem from his first in the special case when $f$ is even. We sketched a proof of this in class, but fill in the details.

(b) Explain why a verbatim adaptation of the proof does not work if $f$ is odd.

(c) Given a general function $f$ and a number $\epsilon > 0$, invoke part (a) to find trigonometric polynomials $P$ and $Q$ that approximate the even functions $f(x) + f(-x)$ and $(f(x) - f(-x))\sin x$ to within $\epsilon$. Now use $P$ and $Q$ to find a trigonometric polynomial $R$ that uniformly approximates $f$ with error at most $\epsilon$, thereby proving Weierstrass’s second theorem.

2. Let $A$ be a normed algebra. If $B$ is a subalgebra of $A$, conclude that $\overline{B}$ is a subalgebra of $A$. Clarification: Recall that for us, an algebra is a vector space equipped with a vector multiplication that is associative, left and right distributive and compatible with scalars.

3. Given a metric space $(X, d)$, recall that $\mathcal{B}(X)$ is the space of all bounded real-valued functions on $X$. Let $A$ be a vector subspace of of $\mathcal{B}(X)$. Show that $A$ is a sublattice of $\mathcal{B}(X)$ if and only if $|f| \in A$.

4. For the examples below, explain whether the class of functions $\mathcal{A}$ is dense in $\mathcal{C}(X)$.

(a) $X = U \times V$, where $U$ and $V$ are compact metric spaces; $\mathcal{A}$ = the class of functions of the form $f(u, v) = g(u)h(v)$ where $g \in \mathcal{C}(U), h \in \mathcal{C}(V)$.

(b) $X$ = a compact set in $\mathbb{R}^n$; $\mathcal{A}$ = the class of all polynomials in $n$-variables.

5. Let $X = \{z : |z| = 1\}$ be the unit circle in the complex plane.

(a) Verify that the space of functions

$$\mathcal{A} = \left\{ f : f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}, \quad \theta \in [0, 2\pi), \quad c_n \in \mathbb{R} \right\}$$

is an algebra.

(b) Show that $\mathcal{A}$ separates points in $X$ and vanishes at no point of $X$.

(c) Show that there exist continuous functions on $X$ that cannot be in the uniform closure of $\mathcal{A}$.

(d) Explain why the statements above do not contradict the Stone-Weierstrass theorem.