1. Let $B[0, 1]$ denote the space of all real-valued bounded functions on $[0, 1]$, equipped with the metric topology generated by the sup norm. Show that $B[0, 1]$ is not separable.

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that $\{f_n : n \geq 1\}$ converges uniformly on the set $\mathbb{Q}$ of rationals. Show that $\{f_n : n \geq 1\}$ actually converges uniformly on all of $\mathbb{R}$. (Hint: First convince yourself that it suffices to show that $\{f_n\}$ is uniformly Cauchy and then prove it.)

3. Given a set $X$, let $B(X)$ denote the vector space of all bounded real-valued functions $f : X \to \mathbb{R}$. Let us endow $B(X)$ with the sup norm

$$||f||_\infty = \sup_{x \in X} |f(x)|,$$

and its accompanying metric topology.

   (a) Verify that $B(X)$ is complete. In other words, if $\{f_n\}$ is a Cauchy sequence in $B(X)$, then show that $\{f_n\}$ converges uniformly to some $f \in B(X)$. Moreover, show that

$$\sup_n ||f_n||_\infty < \infty \quad \text{and} \quad ||f_n||_\infty \to ||f||_\infty \text{ as } n \to \infty.$$

   (b) Let $\{g_n\}$ be a sequence in $B(X)$ satisfying $\sum_{n=1}^{\infty} ||g_n||_\infty < \infty$. Show that $\sum_{n=1}^{\infty} g_n$ converges in $B(X)$ and that

$$||\sum_{n=1}^{\infty} g_n||_\infty \leq \sum_{n=1}^{\infty} ||g_n||_\infty.$$  

   This result is often referred to as the Weierstrass $M$-test.

4. For the series in each of the following examples, determine whether the convergence is uniform, pointwise or neither on the specified interval, with adequate justification.

   (a) the series

$$\sum_{n=1}^{\infty} a_n \sin(nx) \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \cos(nx) \text{ on } \mathbb{R},$$

where $\sum_{n=1}^{\infty} |a_n| < \infty$.

   (b) the series

$$\sum_{n=1}^{\infty} \frac{x^2/(1 + x^2)^n}{|x|^n} \text{ on } |x| \leq 1.$$
5. Recall the space-filling curve we constructed in class, namely

\[ x(t) = \sum_{k=0}^{\infty} 2^{-k-1} f(3^{2k}t) \quad y(t) = \sum_{k=0}^{\infty} 2^{-k-1} f(3^{2k+1}t), \]

where \( f \) is the continuous, piecewise linear, even and 2-periodic function that assumes the value 0 on \([0, \frac{1}{3}]\) and 1 on \([\frac{2}{3}, 1]\). Recall also the definition of the standard middle-third Cantor set \( \Delta \),

\[ \Delta = \bigcap_{n=0}^{\infty} \Delta_n, \]

where \( \Delta_0 = [0, 1] \), and \( \Delta_n \) consists of \( 2^n \) disjoint closed subintervals obtained by removing the middle-third of the intervals that constitute \( \Delta_{n-1} \). Show that \( \{(x(t), y(t)) : t \in \Delta\} = [0, 1] \times [0, 1] \).