1. Give complete definitions of the following terms:

(a) an equicontinuous family of functions in $C[0, 1]$.

Solution. A family of functions $F \subseteq C[0, 1]$ is said to be equicontinuous if for every $\epsilon > 0$, there exists $\delta > 0$ such that
\[ |f(x) - f(y)| < \epsilon \text{ whenever } |x - y| < \delta, \ x, y \in [0, 1] \text{ and for all } f \in F. \]

(b) a sublattice of $B[0, 1]$.

Solution. The space $B[0, 1]$ is equipped with a partial order $\leq$ defined as follows: given $f, g \in B[0, 1]$, we say that $f \leq g$ if for every $x \in [0, 1]$, the real number $f(x)$ is less than or equal to the real number $g(x)$. Given $f, g \in B[0, 1]$, we define functions $m, M \in B[0, 1]$ as follows:
\[ m(x) = \min[f(x), g(x)], \quad M(x) = \max[f(x), g(x)]. \]

These functions have the following properties: if $h, k \in B[0, 1]$ are such that $f, g \leq h$ and $k \leq f, g$, then we must have that $k \leq m$ and $M \leq h$. We refer to $m$ and $M$ as $\min(f, g)$ and $\max(f, g)$ respectively.

We say that $L \subseteq B[0, 1]$ is a sublattice if for every $f, g \in L$, the functions $\min(f, g)$ and $\max(f, g)$ also lie in $L$. □

2. Give examples of the following:

(a) Function classes $F, G \subseteq C[0, 1]$ such that both $F$ and $G$ consist of infinitely many non-constant functions and are uniformly bounded, for which $F$ is equicontinuous but $G$ is not.

Solution. Consider the following subsets of $C[0, 1]$:
\[ F = \{ f_n(x) = x + \frac{1}{n} : n \in \mathbb{N} \}, \quad G = \{ g_n(x) = x^n : n \in \mathbb{N} \}. \]

Both collections are uniformly bounded, $F$ by 2 and $G$ by 1.

The collection $F$ is equicontinuous, since (1) holds with $\delta = \epsilon$. However $G$ is not equicontinuous. We can see this in two ways. Consider $x_0 = 1$. Aiming for a contradiction, suppose for any $\epsilon > 0$ there is a $\delta > 0$ such that $|x - 1| < \delta$ implies $|x^n - 1| < \epsilon$ for all $n$. Then choosing $x = 1 - \frac{1}{n}$ for sufficiently large $n > 1/\delta$, we find that $|(1 - (1 - \frac{1}{n})^n) - 1| < \epsilon$. But the left hand side of this inequality converges to $e^{-1}$ as $n \to \infty$, and hence this inequality is false for large $n$ if $\epsilon = \frac{e^{-1}}{2e}$ for example.

Alternatively, observe that $g_n(x)$ converges pointwise to
\[ g(x) = \begin{cases} 
0 & \text{if } x < 1 \\
1 & \text{if } x = 1.
\end{cases} \]

which is not continuous and hence the convergence cannot be uniform and by HW5 Q5 $G$ cannot be equicontinuous. □
(b) A nontrivial sublattice of $C[0,1]$ that is a subspace but not a subalgebra. Here "nontrivial" means that the sublattice must contain at least one non-constant function.

Solution. The class of piecewise linear functions on $C[0,1]$ is a subspace but not an algebra. □

3. Given any function $f \in C(\mathbb{R}^n)$, show that there exists a sequence $\{p_k\}$ of polynomials in $n$ variables that converges uniformly to $f$ on every compact subset of $\mathbb{R}^n$.

Solution. For any $N \geq 1$, the class of polynomials of $n$ variables forms dense subalgebra of $C(B_N)$ where $B_N = \{x \in \mathbb{R}^n : \|x\| \leq N\}$. This follows from the Stone-Weierstrass theorem, since this subalgebra vanishes nowhere (due to the presence of the constant function 1) and separates points (due to the presence of the polynomials $f_j(x_1, \cdots, x_n) = x_j$). Thus there exists a polynomial $p_N$ such that $\sup_{x \in B_N} |f(x) - p_N(x)| < \frac{1}{N}$.

We claim that the sequence $\{p_N\}$ converges uniformly on every compact subset of $\mathbb{R}^n$. Indeed, given any compact set $K$ and $\epsilon > 0$, we find a large enough $N$ such that $\epsilon < 1$. Then for all $k \geq N$,

$$\sup_{x \in K} |p_k(x) - f(x)| \leq \sup_{x \in B_k} |p_k(x) - f(x)| < \frac{1}{k} \leq \frac{1}{N} < \epsilon,$$

which proves the result. □

4. Give brief answers to the following questions. The answer should be in the form of a short proof or an example, as appropriate.

(a) Is it true that any continuous function $f$ in $C[1,2]$ can be uniformly approximated by a sequence of even polynomials, and also by a sequence of odd polynomials?

Solution. Yes. The function $f(\sqrt{x})$ is continuous on $[1,2]$, hence by Weierstrass's approximation theorem there is a polynomial $p$ that is approximates it arbitrarily closely in sup norm. This implies that $f$ is approximated by the even polynomial $p(x^2)$. In order to approximate $f$ by an odd polynomial, note that the continuous function $f(x)/x$ can be approximated by an even polynomial by the first part of this problem. □

(b) Would your answer to part (a) change if $f$ lies in $C[-1,2]$?

Solution. The answer would change, and the statement of part (a) is no longer true. Since even polynomials do not separate the points $x$ and $-x$, a function such as $\sin x$ would not be approximable by even polynomials. On the other hand, odd polynomials must vanish at zero, hence $\cos x$ cannot be approximated by odd polynomials. □

(c) Let $\{f_n : n \geq 1\}$ be a sequence in $C[a,b]$ with $\|f_n\|_\infty \leq 1$ for all $n$. Define

$$F_n(x) = \int_a^x f_n(t) \, dt.$$

Does $\{F_n\}$ have a uniformly convergent subsequence?
Solution. Yes. The set \( \{ F_n \} \) is uniformly bounded by \( (b-a) \) (using the bound on \( \| f_n \|_\infty \)) and consists of Lipschitz functions with Lipschitz constant 1,

\[
|F_n(x) - F_n(y)| = \left| \int_x^y f_n(t) \, dt \right| \leq |x - y|
\]

hence equicontinuous. By the Arzela-Ascoli theorem, this collection of functions is relatively compact in \( C[a,b] \), hence admits a uniformly convergent subsequence. \( \square \)

(d) Can a sequence of Riemann integrable functions on \([a,b]\) converge pointwise to a non-integrable function?

Solution. Yes. Let \( Q = \{ q_1, q_2, \ldots, q_n, \ldots \} \), and set \( f_n = \chi_{\{q_1, \ldots, q_n\}} \). Each \( f_n \) is Riemann-integrable (with integral 0), and converges pointwise to \( \chi_Q \) which is not. \( \square \)

(e) Consider the space \( C_0(\mathbb{R}) \) of all continuous functions “vanishing at infinity”

\[
C_0(\mathbb{R}) = \{ f \in C(\mathbb{R}) : \lim_{|x| \to \infty} f(x) = 0 \},
\]

endowed with the sup norm. Is this space separable?

Solution. Yes, the space is separable. For each \( N \geq 1 \), let \( G_N \) denote the space of polygonal functions on \([-N,N]\) with rational nodes which vanish on \( |x| \geq N \). We have proved in class that each \( G_N \) is countable. Further given any \( g \in C[-N,N] \) such that \( |g(\pm N)| < \kappa \), we can use the usual uniform continuity argument to find \( G \in G_N \) such that sup\( |x| \leq N \) \( |g(x) - G(x)| < \kappa \).

We claim that the countable union of these sets \( G = \bigcup_{N=1}^{\infty} G_N \), which is countable, is dense in \( C_0(\mathbb{R}) \). To see this fix \( \epsilon > 0 \) and any \( f \in C_0(\mathbb{R}) \). Now choose \( N \) such that \( |f(x)| < \epsilon \) for \( |x| \geq N \). Then pick \( G \in G_N \) such that \( G \) approximates \( f \) restricted to \([-N,N]\) in the sense described above, i.e., sup\( |x| \leq N \) \( |f(x) - G(x)| < \epsilon \). Combining the two inequalities we get that \( \|f-G\|_\infty = \max[\sup_{|x| \leq N} |f(x) - G(x)|, \sup_{|x| \geq N} |f(x)|] < \epsilon \), as desired. \( \square \)