Homework 7 Solutions - Math 321, Spring 2015

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1. *Recall Jordan's theorem: a function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if f can be written as the difference of two non-decreasing functions g and h.*
   (a) *Show that the decomposition $f = g - h$ is by no means unique, and that there are uncountably many ways of writing f in this form.*
   (b) *The following decomposition of f is often useful. Define the positive and negative variations of f by*

$$p(x) = \frac{1}{2}\big(v(x) + f(x) - f(a)\big), \qquad n(x) = \frac{1}{2}\big(v(x) - f(x) + f(a)\big),$$

   where $v(x) = V_a^x f$ is the variation function defined in class. Show that p and n are nondecreasing functions on $[a, b]$ and use this to give an alternative representation of f as the difference of nondecreasing functions.
   (c) *The relevance of p and n is that it injects a certain amount of uniqueness into the Jordan decomposition of f, in the following sense. If g and h are any two non-decreasing functions on $[a, b]$ such that $f = g - h$, then*

$$V_x^y p \le V_x^y g \quad \text{and} \quad V_x^y n \le V_x^y h \text{ for all } x < y \text{ in } [a, b].$$

   Prove this.

Solution. If $f = g-h$ is a decomposition of the appropriate type, then so is $f = (g+c)-(h+c)$, where c is any constant. Thus the decomposition, without further specifications, is highly non-unique.

Given any $x \in [a, b]$, and a partition $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = x\}$ of $[a, x]$, let us define

$$V_+(f, P) = \sum_{i=1}^{n} \max\big[f(x_i) - f(x_{i-1}), 0\big] \quad \text{and} \quad V_-(f, P) = -\sum_{i=1}^{n} \min\big[f(x_i) - f(x_{i-1}), 0\big].$$

It is clear that $V_\pm(f, P) \ge V_\pm(f, P')$ whenever P is a refinement of P', and that

$$V(f, P) = V_+(f, P) + V_-(f, P), \qquad f(x) - f(a) = V_+(f, P) - V_-(f, P)$$

for any partition P. Setting

$$v_+(x) = \sup_P V(f, P), \qquad v_-(x) = \sup_P V_-(f, P),$$

and choosing increasingly finer partitions P of $[a, x]$, we find that

$$v_+(x) + v_-(x) = v(x) \quad \text{and} \quad v_+(x) - v_-(x) = f(x) - f(a).$$

Hence $p(x) = v_+(x)$ and $n(x) = v_-(x)$. Since $v_\pm$ are both non-decreasing functions of x by definition, the result in part (b) follows.

Suppose now that g, h are nondecreasing functions such that $f = g - h$. Then for any $x < y$ in $[a, b]$,

$$
\begin{aligned}
V_x^y p = v_+(y) - v_+(x) &= \sup_Q \sum_{i=1}^n \max\big[f(q_i) - f(q_{i-1}), 0\big] \\
&= \sup_Q \sum_{i=1}^n \max\big[g(q_i) - g(q_{i-1}) - \big(h(q_i) - h(q_{i-1})\big), 0\big] \\
&\le \sup_Q \sum_{i=1}^n \max\big[g(q_i) - g(q_{i-1}), 0\big] = \sup_Q \sum_{i=1}^n \big[g(q_i) - g(q_{i-1})\big] \\
&= V_x^y g.
\end{aligned}
$$

Here $Q = \{x = q_0 < q_1 < \cdots < q_n = y\}$ denotes an arbitrary partition of $[x, y]$. The inequality for $V_x^y n$ and $V_x^y h$ is similarly obtained.

□

2. *We stated the "integration by parts" formula in class, using it to highlight the interchangability of integrand and integrator. The purpose of this problem is to fill in the details of its proof. Throughout this problem f and α denote arbitrary real-valued functions on $[a, b]$.*

   (a) *Given any partition $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ and a collection of points $T = \{t_1, \cdots, t_n\}$ with $t_j \in [x_{j-1}, x_j]$, prove the following identity:*

   $$S_f(\alpha, P, T) = f(b)\alpha(b) - f(a)\alpha(a) - S_\alpha(f, P', T').$$

   Here $P' = \{a = t_0, t_1, \cdots, t_n, t_{n+1} = b\}$ and $T' = P$.

   (b) *Use part (a) to show that $f \in \mathcal{R}_\alpha[a, b]$ if and only if $\alpha \in \mathcal{R}_f[a, b]$. Show that in either case,*

   $$\int_a^b f\, d\alpha + \int_a^b \alpha\, df = f(b)\alpha(b) - f(a)\alpha(a).$$

   Note that this is one of those rare instances where one implication implies the other!

Solution. For any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and any choice of points $T = \{t_1, \cdots, t_n\}$ with $t_i \in I_i = [xi - 1, x_i]$, we can arrange the sum representing $S_f(\alpha, P, T)$ as follows:

$$
\begin{aligned}
S_f(\alpha, P, T) &= \sum_{i=1}^n \alpha(t_i)\,[f(x_i) - f(x_{i-1})] \\
&= \sum_{i=1}^n f(x_i)\alpha(t_i) - \sum_{i=1}^n \alpha(t_i) f(x_{i-1}) \\
&= \sum_{i=1}^n f(x_i)\alpha(t_i) - \sum_{i=0}^{n-1} f(x_i)\alpha(t_{i+1}) \\
&= -\sum_{i=0}^n f(x_i)\,[\alpha(t_{i+1}) - \alpha(t_i)] - f(x_0)\alpha(t_0) + f(x_n)\alpha(t_{n+1}) \\
&= f(b)\alpha(b) - f(a)\alpha(a) - S_\alpha(f, P', T').
\end{aligned}
$$

This completes the proof of part (a).

For part (b), let us assume that $f \in \mathcal{R}_\alpha[a,b]$ and fix $\epsilon > 0$. We choose a parititon P^* so that

$$\left|S_\alpha(f,P,T) - \int_a^b f\,d\alpha\right| < \epsilon \text{ for all } P \supset P^* \text{ and any choice of } T.$$

Since P' as described above need not be a refinement of P, we consider the natural modification $P'' = \{x_0, t_1, x_1, t_2, \cdots\} \supset P$, with the accompanying choice of points being $T'' = \{x_0, x_0, x_1, x_1, \cdots, x_n, x_n\}$, and observe that

$$\begin{aligned} S_\alpha(f,P',T') &= \sum_{i=0}^n f(x_i)\big[\alpha(t_{i+1}) - \alpha(t_i)\big] \\ &= \sum_{i=0}^n f(x_i)\big[\alpha(t_{i+1}) - \alpha(x_i)\big] + \sum_{i=0}^n f(x_i)\big[\alpha(x_i) - \alpha(t_i)\big] \\ &= S_\alpha(f,P'',T''). \end{aligned}$$

Now for any $P \supset P^*$, we appeal to part (a) of the problem, obtaining

$$\left|S_f(\alpha,P,T) - \left[f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b f\,d\alpha\right]\right| = \left|\int_a^b f\,d\alpha - S_\alpha(f,P'',T'')\right| < \epsilon.$$

Since ϵ is arbitrary, this shows that $\alpha \in \mathcal{R}_f[a,b]$ and

$$\int_a^b \alpha\,df = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b f\,d\alpha.$$

Note: It is not enough just to find a partition P^* satisfying

$$\left|S_f(\alpha,P^*,T) - \left[f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b f\,d\alpha\right]\right| < \epsilon$$

We also need its refinements to satisfy this. □

3. Let $\alpha \in \mathrm{BV}[a,b]$ and let $\beta(x) = V_a^x\alpha$. Recall that both β and $\beta - \alpha$ are increasing. Show that $\mathcal{R}_\alpha[a,b] = \mathcal{R}_\beta[a,b] \cap \mathcal{R}_{\beta-\alpha}[a,b]$. This identity was instrumental to our conclusion that $\mathcal{R}_\alpha[a,b]$ is a vector space, an algebra and a lattice. (*Hint:* Argue that it suffices to only show that $\mathcal{R}_\alpha[a,b] \subseteq \mathcal{R}_\beta[a,b]$.)

Solution. Let $\beta(x) = V_a^x\alpha$. First we show that $R_\alpha[a,b] \subseteq R_\beta[a,b]$. Let $f \in R_\alpha[a,b], \epsilon > 0$ then $\exists I, P^*, \forall P \supset P^*$ and choice of T corresponding to P, we have

$$|S_\alpha(f,P,T) - I| < \epsilon$$

Pick $P = \{a = x_0 < \cdots < x_n = b\} \supset P^*$ such that $|V(\alpha,P) - V_a^b\alpha| < \epsilon$. Let,

$$M_i = \sup_{x\in[x_{i-1},x_i]} f(x),\ m_i = \inf_{x\in[x_{i-1},x_i]} f(x),\ \Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

Now pick two sets of sample points $S = \{s_1, \cdots, s_n\}, T = \{t_1, \cdots, t_n\}$ where $s_i, t_i \in [x_{i-1}, x_i]$ such that if $\Delta\alpha_i \geq 0$,

$$M_i - \frac{\epsilon}{2} < f(s_i) \leq M_i, \quad m_i \leq f(t_i) < m_i + \frac{\epsilon}{2}$$

and if $\Delta\alpha_i < 0$,

$$M_i - \frac{\epsilon}{2} < f(t_i) \leq M_i, \quad m_i \leq f(s_i) < m_i + \frac{\epsilon}{2}$$

This implies

$$(M_i - m_i)|\Delta\alpha_i| \geq (f(s_i) - f(t_i))\Delta\alpha_i$$

Using $\sum_{i=1}^n |\Delta\alpha_i| \leq V_a^b\alpha$ then

$$\sum_{i=1}^n (M_i - m_i)|\Delta\alpha_i| \geq \sum_{i=1}^n (f(s_i) - f(t_i))\Delta\alpha_i > \sum_{i=1}^n (M_i - m_i)|\Delta\alpha_i| - \epsilon V_a^b\alpha$$

Hence

$$|\sum_{i=1}^n (M_i - m_i)|\Delta\alpha_i| - \sum_{i=1}^n (f(s_i) - f(t_i))\Delta\alpha_i| < \epsilon V_a^b f$$

Now by our choice of $P, |S_\alpha(f, P, S) - I| < \epsilon, |S_\alpha(f, P, T) - I| < \epsilon$. Hence

$$|\sum_{i=1}^n (f(s_i) - f(t_i))\Delta\alpha_i| = |S_\alpha(f, P, S) - S_\alpha(f, P, T)| < 2\epsilon$$

Hence

$$\begin{aligned}\left|\sum_{i=1}^n (M_i - m_i)|\Delta\alpha_i|\right| &\leq \left|\sum_{i=1}^n (M_i - m_i)|\Delta\alpha_i| - \sum_{i=1}^n (f(s_i) - f(t_i))\Delta\alpha_i\right| + \left|\sum_{i=1}^n (f(s_i) - f(t_i))\Delta\alpha_i\right| \\ &< \epsilon V_a^b\alpha + 2\epsilon\end{aligned}$$

Now we are ready to show that $f \in R_\beta[a, b]$. Since f is BV it is bounded by some M and so $|M_i - m_i| \leq 2M$. Also

$$\Delta\beta_i = V_{x_{i-1}}^{x_i}\alpha \geq |\alpha(x_i) - \alpha(x_{i-1})| = |\Delta\alpha_i|$$

Hence

$$\begin{aligned}|U_\beta(f, P) - L_\beta(f, P)| &= |\sum_{i=1}^n (M_i - m_i)\Delta\beta_i| \\ &= \left|\sum_{i=1}^n (M_i - m_i)(\Delta\beta_i - |\Delta\alpha_i|)\right| + \left|\sum_{i=1}^N (M_i - m_i)|\Delta\alpha_i|\right| \\ &< \sum_{i=1}^n |M_i - m_i|(\Delta\beta_i - |\Delta\alpha_i|) + (V_a^b\alpha + 2)\epsilon \\ &\leq 2M\sum_{i=1}^n (\Delta\beta_i - |\Delta\alpha_i|) + (V_a^b\alpha + 2)\epsilon \\ &= 2M(V_a^b\alpha - V(\alpha, P)) + (V_a^b\alpha + 2)\epsilon \\ &< 2M\epsilon + (V_a^b\alpha + 2)\epsilon\end{aligned}$$

So $f \in R_\beta[a.b]$.
Finally, using Question 2, $f \in R_\alpha[a, b] \Rightarrow f \in R_\beta[a, b] \Rightarrow \alpha, \beta \in R_f[a, b] \Rightarrow \beta - \alpha \in R_f[a, b]$ $\Rightarrow f \in R_{\beta-\alpha}[a, b] \Rightarrow f \in R_{\beta-\alpha}[a, b] \cap R_\beta[a, b]$.
$f \in R_{\beta-\alpha}[a, b] \cap R_\beta[a, b] \Rightarrow \beta, \alpha - \beta \in R_f[a, b] \Rightarrow \alpha \in R_f[a, b]$. $\Rightarrow f \in R_\alpha[a, b]$. $\square$

4. Let $\{f_n\}$ be a bounded sequence in $\mathrm{BV}[a,b]$, i.e., suppose that $||f_n||_{\mathrm{BV}} \leq K$ for all n. Show that f_n admits a pointwise convergent subsequence whose limit f lies in $\mathrm{BV}[a,b]$ with $||f||_{\mathrm{BV}} \leq K$. This is known as Helly's first theorem. (*Hint:* First try out the case when all the functions f_n are non-decreasing, then adapt it for functions of bounded variation.)

Proof. Assume first that f_n is non-decreasing. Since $\|f_n\|_{BV} \leq K$, we have

$$|f_n(x)| \leq |f_n(x) - f_n(a)| + |f_n(a)| \leq V_a^b f_n + |f_n(a)| = \|f_n\|_{BV} \leq K$$

So f is bounded by K. Hence for each $x \in [a,b]$, $f_n(x)$ has a convergent subsequence. Since $\mathbb{Q}$ is countable. By cantor's diagonal argument we can find a subsequence $g_k = f_{n_k}$ that converges pointwise on $[a,b]$. Define $g : [a,b] \to \mathbb{R}$.

$$g(x) = \begin{cases} \lim_{k\to\infty} g_k(x) & \text{if } x \in \mathbb{Q} \\ \sup_{q<x, q\in\mathbb{Q}\cap[a,b]} g(q) & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Since g_k is nondecreasing, g is nondecreasing on $\mathbb{Q}$ and then by the definition, g is nondecreasing on $[a.b]$.

Claim: g_k actually converges to g at every point that g is continuous.
Proof of claim: Suppose g is continuous at x. Choose $\delta > 0$ such that $|x-y| < \delta \implies |g(x) - g(y)| < \epsilon$. Since $\mathbb{Q}$ is dense we can find $p, q \in \mathbb{Q} \cap [a,b]$ such that $p, q \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]$ and since g_k converges at p, q then if k is sufficiently large,

$$|g_k(p) - g_k(q)| \leq |g_k(p) - g(p)| + |g(p) - g(q)| + |g_k(q) - g(q)| \leq 3\epsilon$$

□

Since g_k is nondecreasing, $g_k(p) \leq g_k(x) \leq g_k(q)$ then

$$\begin{aligned} |g_k(x) - g(x)| &\leq |g_k(x) - g_k(p)| + |g_k(p) - g(p)| + |g(p) - g(x)| \\ &\leq |g_k(q) - g(q)| + |g_k(p) - g(p)| + |g(p) - g(x)| \\ &< 3\epsilon + \epsilon + \epsilon \end{aligned}$$

So we have the claim. Now g is non-decreasing, it has at most a countable number of discontinuities. As g_k is uniformly bounded, we can again use the diagonal argument to extract another subsequence of g_k the converges at every point on $[a,b]$. We are done in case that g_n is nondecreasing.
Now for any $f_n \in BV[a,b], \|f_n\|_{BV} \leq K$ we can decompose $f_n = g_n - h_n$ where $g_n(x) = V_a^x f_n, h_n = g_n - f_n$. we know that both are nondecreasing. We can see

$$|g_n(x)| \leq K, |h_n(x)| \leq |f_n(x)| + |g_n(x)| \leq 2K$$

By the previous case, g_n has a subsequence g_{n_k} that converges at every point in $[a,b]$. Apply the argument again to h_{n_k} then there is a subsequence, call it g_{n_k}, h_{n_k} that both of them converge everywhere in $[a,b]$. i.e. f_{n_k} converges pointwise to some function f on $[a,b]$. We will show that $\|f\|_{BV} \leq K$. Fix a partition P,

$$V(f_{n_k}, P) + |f_{n_k}(a)| \leq V_a^b f_{n_k} + |f_{n_k}(a)| \leq K$$

Since there is only finitely many terms in $V(f_{n_k}, P)$, let $k \to \infty$ then

$$V(f, P) + |f_{n_k}(a)| \leq K$$

Take supremum over P we have $\|f\|_{BV} \leq K$ as required.