Recall Jordan’s theorem: a function \( f : [a, b] \to \mathbb{R} \) is of bounded variation if and only if \( f \) can be written as the difference of two non-decreasing functions \( g \) and \( h \).

(a) Show that the decomposition \( f = g - h \) is by no means unique, and that there are uncountably many ways of writing \( f \) in this form.

(b) The following decomposition of \( f \) is often useful. Define the positive and negative variations of \( f \) by

\[
p(x) = \frac{1}{2} (v(x) + f(x) - f(a)), \quad n(x) = \frac{1}{2} (v(x) - f(x) + f(a)),
\]

where \( v(x) = V_0^x f \) is the variation function defined in class. Show that \( p \) and \( n \) are nondecreasing functions on \([a, b]\) and use this to give an alternative representation of \( f \) as the difference of nondecreasing functions.

(c) The relevance of \( p \) and \( n \) is that it injects a certain amount of uniqueness into the Jordan decomposition of \( f \), in the following sense. If \( g \) and \( h \) are any two non-decreasing functions on \([a, b]\) such that \( f = g - h \), then

\[
V_x^y p \leq V_x^y g \quad \text{and} \quad V_x^y n \leq V_x^y h \quad \text{for all} \ x < y \ \text{in} \ [a, b].
\]

Prove this.

**Solution.** If \( f = g - h \) is a decomposition of the appropriate type, then so is \( f = (g+c)-(h+c) \), where \( c \) is any constant. Thus the decomposition, without further specifications, is highly non-unique.

Given any \( x \in [a, b] \), and a partition \( P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = x\} \) of \([a, x]\), let us define

\[
V_+(f, P) = \sum_{i=1}^{n} \max[f(x_i) - f(x_{i-1}), 0] \quad \text{and} \quad V_-(f, P) = -\sum_{i=1}^{n} \min[f(x_i) - f(x_{i-1}), 0].
\]

It is clear that \( V_+(f, P) \geq V_+(f, P') \) whenever \( P \) is a refinement of \( P' \), and that

\[
V(f, P) = V_+(f, P) + V_-(f, P), \quad f(x) - f(a) = V_+(f, P) - V_-(f, P)
\]

for any partition \( P \). Setting

\[
v_+(x) = \sup_P V(f, P), \quad v_-(x) = \sup_P V_-(f, P),
\]

and choosing increasingly finer partitions \( P \) of \([a, x]\), we find that

\[
v_+(x) + v_-(x) = v(x) \quad \text{and} \quad v_+(x) - v_-(x) = f(x) - f(a).
\]

Hence \( p(x) = v_+(x) \) and \( n(x) = v_-(x) \). Since \( v_\pm \) are both non-decreasing functions of \( x \) by definition, the result in part (b) follows.

Suppose now that \( g, h \) are nondecreasing functions such that \( f = g - h \). Then for any \( x < y \) in \([a, b]\),
\[ V_x^y p = v_+(y) - v_+(x) = \sup_Q \sum_{i=1}^{n} \max[f(q_i) - f(q_{i-1}), 0] \]
\[ = \sup_Q \sum_{i=1}^{n} \max[g(q_i) - g(q_{i-1}) - (h(q_i) - h(q_{i-1}))], 0] \]
\[ \leq \sup_Q \sum_{i=1}^{n} \max[g(q_i) - g(q_{i-1}), 0] = \sup_Q \sum_{i=1}^{n} [g(q_i) - g(q_{i-1})] \]
\[ = V_x^y g. \]

Here \( Q = \{x = q_0 < q_1 < \cdots < q_n = y\} \) denotes an arbitrary partition of \([x,y]\). The inequality for \( V_x^y n \) and \( V_x^y h \) is similarly obtained.

\[ \square \]

2. We stated the “integration by parts” formula in class, using it to highlight the interchanga-
bility of integrand and integrator. The purpose of this problem is to fill in the details of its
proof. Throughout this problem \( f \) and \( \alpha \) denote arbitrary real-valued functions on \([a,b]\).

(a) Given any partition \( P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\} \) and a collection of points
\( T = \{t_1, \cdots, t_n\} \) with \( t_k \in [x_{j-1}, x_j] \), prove the following identity:

\[ S_f(\alpha, P, T) = f(b)\alpha(b) - f(a)\alpha(a) - S_\alpha(f, P', T'). \]

Here \( P' = \{a = t_0, t_1, \cdots, t_n, t_{n+1} = b\} \) and \( T' = P \).

(b) Use part (a) to show that \( f \in \mathcal{R}_\alpha[a,b] \) if and only if \( \alpha \in \mathcal{R}_f[a,b] \). Show that in either
case,

\[ \int_a^b f \, d\alpha + \int_a^b \alpha \, df = f(b)\alpha(b) - f(a)\alpha(a). \]

Note that this is one of those rare instances where one implication implies the other!

**Solution.** For any partition \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \) and any choice of points
\( T = \{t_1, \cdots, t_n\} \) with \( t_i \in I_i = [x_i - 1, x_i] \), we can arrange the sum representing \( S_f(\alpha, P, T) \)
as follows:

\[ S_f(\alpha, P, T) = \sum_{i=1}^{n} \alpha(t_i) [f(x_i) - f(x_{i-1})] \]
\[ = \sum_{i=1}^{n} f(x_i)\alpha(t_i) - \sum_{i=1}^{n} \alpha(t_i) f(x_{i-1}) \]
\[ = \sum_{i=1}^{n} f(x_i)\alpha(t_i) - \sum_{i=0}^{n-1} f(x_i)\alpha(t_{i+1}) \]
\[ = -\sum_{i=0}^{n} f(x_i) \left[ \alpha(t_{i+1}) - \alpha(t_i) \right] - f(x_0)\alpha(t_0) + f(x_n)\alpha(t_{n+1}) \]
\[ = f(b)\alpha(b) - f(a)\alpha(a) - S_\alpha(f, P', T'). \]
3. Let $\alpha$ such that if $\Delta R$ that $R$ then $\exists I, P$, we have $\alpha, P, T$.

We also need its refinements to satisfy this.

Now pick two sets of sample points $S = \{a, b\} \subseteq R[a, b]$. This identity was instrumental to our conclusion that $R[a, b]$ is a vector space, an algebra and a lattice. (Hint: Argue that it suffices to only show that $\mathcal{R}[a, b] \subseteq R[a, b].$)

Solution. Let $\beta(x) = V^x\alpha$. First we show that $R[a, b] \subseteq R[\beta]$. Let $P \in R[a, b], \epsilon > 0$ then $\exists I, P, \forall P \supset P^*$ and choice of $T$ corresponding to $P$, we have $|S(f, P, T) - I| < \epsilon$

Pick $P = \{a = x_0 < \cdots < x_n = b\} \supset P^*$ such that $|V(\alpha, P) - V^b\alpha| < \epsilon$. Let,

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x), \ m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \ \Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

Now pick two sets of sample points $S = \{s_1, \cdots, s_n\}, T = \{t_1, \cdots, t_n\}$ where $s_i, t_i \in [x_{i-1}, x_i]$ such that if $\Delta \alpha_i \geq 0$,

$$M_i - \frac{\epsilon}{2} < f(s_i) \leq M_i, \ m_i \leq f(t_i) < m_i + \frac{\epsilon}{2}$$
and if $\Delta \alpha_i < 0$,

$$M_i - \frac{\epsilon}{2} < f(t_i) \leq M_i, \quad m_i \leq f(s_i) < m_i + \frac{\epsilon}{2}$$

This implies

$$(M_i - m_i)|\Delta \alpha_i| \geq (f(s_i) - f(t_i))\Delta \alpha_i$$

Using $\sum_{i=1}^{n} |\Delta \alpha_i| \leq V_a^b \alpha$ then

$$\sum_{i=1}^{n} (M_i - m_i)|\Delta \alpha_i| \geq \sum_{i=1}^{n} (f(s_i) - f(t_i))\Delta \alpha_i > \sum_{i=1}^{n} (M_i - m_i)|\Delta \alpha_i| - \epsilon V_a^b \alpha$$

Hence

$$\left|\sum_{i=1}^{n} (M_i - m_i)|\Delta \alpha_i| - \sum_{i=1}^{n} (f(s_i) - f(t_i))\Delta \alpha_i\right| < \epsilon V_a^b \alpha$$

Now by our choice of $P$, $|S_\alpha(f, P, S) - I| < \epsilon, |S_\alpha(f, P, T) - I| < \epsilon$. Hence

$$\left|\sum_{i=1}^{n} (f(s_i) - f(t_i))\Delta \alpha_i\right| = |S_\alpha(f, P, S) - S_\alpha(f, P, T)| < 2\epsilon$$

Hence

$$\left|\sum_{i=1}^{n} (M_i - m_i)|\Delta \alpha_i|\right| \leq \left|\sum_{i=1}^{n} (M_i - m_i)|\Delta \alpha_i| - \sum_{i=1}^{n} (f(s_i) - f(t_i))\Delta \alpha_i\right| + \left|\sum_{i=1}^{n} (f(s_i) - f(t_i))\Delta \alpha_i\right|$$

$$< \epsilon V_a^b \alpha + 2\epsilon$$

Now we are ready to show that $f \in R_\beta[a, b]$. Since $f$ is BV it is bounded by some $M$ and so $|M_i - m_i| \leq 2M$. Also

$$\Delta \beta_i = V_{x_{i-1}} \alpha \geq |\alpha(x_i) - \alpha(x_{i-1})| = |\Delta \alpha_i|$$

Hence

$$|U_\beta(f, P) - L_\beta(f, P)| = \left|\sum_{i=1}^{n} (M_i - m_i)\Delta \beta_i\right|$$

$$= \left|\sum_{i=1}^{n} (M_i - m_i)(\Delta \beta_i - |\Delta \alpha_i|)\right| + \left|\sum_{i=1}^{n} (M_i - m_i)|\Delta \alpha_i|\right|$$

$$< \sum_{i=1}^{n} |M_i - m_i|(|\Delta \beta_i - |\Delta \alpha_i|) + (V_a^b \alpha + 2\epsilon)$$

$$\leq 2M \sum_{i=1}^{n} (|\Delta \beta_i - |\Delta \alpha_i|) + (V_a^b \alpha + 2\epsilon)$$

$$= 2M(V_a^b \alpha - V(\alpha, P)) + (V_a^b \alpha + 2\epsilon)$$

$$< 2M\epsilon + (V_a^b \alpha + 2\epsilon)$$

So $f \in R_\beta[a, b]$. Finally, using Question 2, $f \in R_\alpha[a, b] \Rightarrow f \in R_\beta[a, b] \Rightarrow \alpha, \beta \in R_f[a, b] \Rightarrow \beta - \alpha \in R_f[a, b] \Rightarrow f \in R_{\beta - \alpha}[a, b]$.

$f \in R_{\beta - \alpha}[a, b] \Rightarrow f \in R_{\beta - \alpha}[a, b] \cap R_\beta[a, b]$. 

$f \in R_{\beta - \alpha}[a, b] \cap R_\beta[a, b] \Rightarrow \beta, \alpha - \beta \in R_f[a, b] \Rightarrow \alpha \in R_f[a, b] \Rightarrow f \in R_\alpha[a, b]$. 

\qed
4. Let \( \{f_n\} \) be a bounded sequence in \( BV[a, b] \), i.e., suppose that \( \|f_n\|_{BV} \leq K \) for all \( n \). Show that \( f_n \) admits a pointwise convergent subsequence whose limit \( f \) lies in \( BV[a, b] \) with \( \|f\|_{BV} \leq K \). This is known as Helly’s first theorem. (Hint: First try out the case when all the functions \( f_n \) are non-decreasing, then adapt it for functions of bounded variation.)

Proof. Assume first that \( f_n \) is non-decreasing. Since \( \|f_n\|_{BV} \leq K \), we have
\[
|f_n(x)| \leq |f_n(x) - f_n(a)| + |f_n(a)| \leq V^b_a f_n + |f_n(a)| = \|f_n\|_{BV} \leq K
\]
So \( f \) is bounded by \( K \). Hence for each \( x \in [a, b] \), \( f_n(x) \) has a convergent subsequence. Since \( Q \) is countable. By cantor’s diagonal argument we can find a subsequence \( g_k = f_{n_k} \) that converges pointwise on \([a, b]\). Define \( g : [a, b] \to \mathbb{R} \).

\[
g(x) = \begin{cases} 
\lim_{k \to \infty} g_k(x) & \text{if } x \in Q \\
\sup_{q < x, q \in Q \cap [a, b]} g(q) & \text{if } x \notin Q.
\end{cases}
\]

Since \( g_k \) is nondecreasing, \( g \) is nondecreasing on \( Q \) and then by the definition, \( g \) is nondecreasing on \([a, b]\).

**Claim:** \( g_k \) actually converges to \( g \) at every point that \( g \) is continuous.

Proof of claim: Suppose \( g \) is continuous at \( x \). Choose \( \delta > 0 \) such that \( |x - y| < \delta \implies |g(x) - g(y)| < \epsilon \). Since \( Q \) is dense we can find \( p, q \in Q \cap [a, b] \) such that \( p, q \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}] \) and since \( g_k \) converges at \( p, q \) then if \( k \) is sufficiently large,
\[
|g_k(p) - g_k(q)| \leq |g_k(p) - g(p)| + |g(p) - g(q)| + |g_k(q) - g(q)| \leq 3\epsilon
\]

Since \( g_k \) is nondecreasing, \( g_k(p) \leq g_k(x) \leq g_k(q) \) then
\[
|g_k(x) - g(x)| \leq |g_k(x) - g_k(p)| + |g_k(p) - g(p)| + |g(p) - g(x)|
\]
\[
\leq |g_k(q) - g(q)| + |g_k(p) - g(p)| + |g(p) - g(x)|
\]
\[
< 3\epsilon + \epsilon + \epsilon
\]
So we have the claim. Now \( g \) is non-decreasing, it has at most a countable number of discontinuities. As \( g_k \) is uniformly bounded, we can again use the diagonal argument to extract another subsequence of \( g_k \) the converges at every point on \([a, b]\). We are done in case that \( g_n \) is nondecreasing.

Now for any \( f_n \in BV[a, b], \|f_n\|_{BV} \leq K \) we can decompose \( f_n = g_n - h_n \) where \( g_n(x) = V^x_a f_n, h_n = g_n - f_n \). we know that both are nondecreasing. We can see
\[
|g_n(x)| \leq K, |h_n(x)| \leq |f_n(x)| + |g_n(x)| \leq 2K
\]
By the previous case, \( g_n \) has a subsequence \( g_{n_k} \) that converges at every point in \([a, b]\). Apply the argument again to \( h_{n_k} \) then there is a subsequence, call it \( g_{n_{k}}, h_{n_{k}} \) that both of them converge everywhere in \([a, b]\). i.e. \( f_{n_{k}} \) converges pointwise to some function \( f \) on \([a, b]\). We will show that \( \|f\|_{BV} \leq K \). Fix a partition \( P, \)
\[
\|f\|_{BV} \leq K
\]
Since there is only finitely many terms in \( V(f_{n_{k}}, P) \), let \( k \to \infty \) then
\[
V(f, P) + |f_{n_{k}}(a)| \leq K
\]
Take supremum over \( P \) we have \( \|f\|_{BV} \leq K \) as required.