Homework 6 Solutions - Math 321, Spring 2015

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1. Given a nonconstant non-decreasing function $\alpha : [a, b] \to \mathbb{R}$, let $\mathcal{R}_\alpha[a, b]$ denote the collection of all bounded functions on $[a, b]$ which are Riemann-Stieltjes integrable with respect to α. Is $\mathcal{R}_\alpha[a, b]$ a vector space, a lattice, an algebra?

   Solution. They are all a vector space, a lattice, an algebra. (See also Theorem 6.12-6.13 in the textbook.) To show that $f, g \in \mathcal{R}_\alpha[a, b] \implies f + g, cf, |f| \in \mathcal{R}_\alpha[a, b]$, we use (verify it !)

$$\begin{aligned} U_\alpha(f+g, P) - L_\alpha(f+g, P) &\le (U_\alpha(f, P) - L_\alpha(f, P)) + (U_\alpha(g, P) - L_\alpha(g, P)) \\ U_\alpha(cf, P) - L_\alpha(cf, P) &= cU_\alpha(f, P) - cL_\alpha(g, P) \\ U_\alpha(|f|, P) - L_\alpha(|g|, P) &\le U_\alpha(f, P) - L_\alpha(g, P) \end{aligned}$$

   To show that $fg \in \mathcal{R}_\alpha[a, b]$. First we prove in the special case that f, g are non-negative then general case follows by decomposing $f = f^+ - f^-, g = g^+ - g^-$ and we know that $f^+, f^-, g^+, g^- \in \mathcal{R}_\alpha[a, b]$ since it is a lattice.
   Now assume that f, g are non-negative, we use

$$\begin{aligned} |U_\alpha(fg, P) - L_\alpha(fg, P)| &\le |U_\alpha(f, P)U_\alpha(g, P) - L_\alpha(f, P)L_\alpha(g, P)| \\ &\le |U_\alpha(f, P)||U_\alpha(g, P) - L_\alpha(g, P)| + |L_\alpha(g, P)||U_\alpha(f, P) - L_\alpha(f, P)| \end{aligned}$$

   Here $|U_\alpha(f, P)|, |L_\alpha(g, P)|$ are bounded by an absolute constant. $\square$

2. This problem focuses on computing the Riemann-Stieltjes integral for specific choices of integrators.

   (a) Let $x_0 = a < x_1 < x_2 < \cdots < x_n = b$ be a fixed collection of points in $[a, b]$, and let α be an increasing step function on $[a, b]$ that is constant on each of the open intervals (x_{i-1}, x_i) and has jumps of size $\alpha_i = \alpha(x_i+) - \alpha(x_i-)$ at each of the points x_i. For $i = 0$ and n, we make the obvious adjustments

$$\alpha_0 = \alpha(a+) - \alpha(a-), \qquad \alpha_n = \alpha(b) - \alpha(b-).$$

   If $f \in B[a, b]$ is continuous at each of the points x_i, show that $f \in \mathcal{R}_\alpha[a, b]$ and

$$\int_a^b f\, d\alpha = \sum_{i=0}^{n} f(x_i)\alpha_i.$$

   Solution. Intuitively, the point x_i will contribute the value of the integral if α jumps at x_i which α_i measures how large does α jump. . Given $\epsilon > 0$, for each x_i there is a δ_i such that $|x - x_i| < \delta \implies |f(x) - f(x_i)| < \epsilon$. Choose $\delta = \min\{\delta_1, \ldots, \delta_n\}$ then $|f(x) - f(x_i)| < \epsilon$. (or one can argue that f is uniformly continuous). Also assume $\delta < \frac{1}{2} \min_{1 \le i \le n} |x_i - x_{i+1}|$.
   Refine the partition P as follows:
   For each $x_0, x_1, \ldots, x_{n-1}$ choose a point $y_i \in (x_i, x_i + \delta)$.
   For each $x_1, x_2, \ldots, x_n$ choose a point $z_i \in (x_i - \delta, x_i)$. Then we have a new partition $P*$

$$\{x_0, y_0, z_1, x_1, y_1, z_2, x_2, y_2, \ldots, z_{n-1}, x_{n-1}, y_{n-1}, z_n, x_n\}$$

   $\square$

Now using definition of δ and the fact that α is constant along (x_i, x_{i+1})

$$L_\alpha(f, P^*) > \sum_{i=1}^{n} (f(x_i) - \epsilon)\alpha_i$$

$$U_\alpha(f, P^*) < \sum_{i=1}^{n} (f(x_i) + \epsilon)\alpha_i$$

We have

$$\sum_{i=1}^{n} (f(x_i) - \epsilon)\alpha_i < L_\alpha(f, P^*) < \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha < U_\alpha(f, P^*) < \sum_{i=1}^{n} (f(x_i) + \epsilon)\alpha_i$$

Since $\epsilon > 0$ is arbitrary (small), let $\epsilon \to 0$, we are done.

(b) If f is continuous on $[1, n]$, compute $\int_1^n f(x) d[x]$, where $[x]$ is the greatest integer in x. What is the value of $\int_1^t f(x) d[x]$ if t is not an integer?

Solution. Apply part a) with $x_0 = 1, x_1 = 2, \ldots, x_{n-1} = n$ since $\alpha_0 = 0, \alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 1$, we have

$$\int_1^n f d[x] = \sum_{i=0}^{n-1} f(n)\alpha_i = \sum_{k=2}^{n} f(k)$$

i.e. on $[1, N], [x]$ jumps at $2, 3, \ldots, N$. In the same way,

$$\int_1^t f d[x] = \sum_{k=2}^{[t]} f(k)$$

In we want the sum from 1 to n we could take e.g.

$$\int_{0.99}^t f d[x] = \sum_{k=1}^{[t]} f(k)$$

where 0.99 could be replaced by any numbers in $[0, 1)$. $\square$

3. Determine, with adequate justification, whether each of the following statements is true or false.

(a) *An equicontinuous, pointwise bounded subset of $\mathcal{C}[a, b]$ is compact.*

Solution. The statement is easily seen to be false; the class of constant functions $\{\frac{1}{n} : n \geq 1\}$ provides a counterexample. The collection is equicontinuous but not closed (the constant function zero is a limit point outside the set), hence not compact.
Remark: Note however that an equicontinuous, pointwise bounded and *closed* subset of $\mathcal{C}(X)$ is in fact compact. Just follow the proof of the Arzela-Ascoli theorem. $\square$

(b) *The function $\chi_{\mathbb{Q}}$ is Riemann integrable on $[0, 1]$.*

Solution. The function $\chi_{\mathbb{Q}}$ is not Riemann integrable on $[0, 1]$ or on any interval $[a, b]$ for that matter. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, it follows that $U(\chi_{\mathbb{Q}}, P) = 1$ and $L(\chi_{\mathbb{Q}}) = 0$ for any partition P of $[a, b]$. By Riemann's condition on integrability, $\chi_{\mathbb{Q}}$ fails to be integrable. $\square$

(c) *The function χ_Δ is Riemann integrable on $[0,1]$, where Δ denotes the Cantor middle-third set. (We have already run into this set in Homework 2, Problem 5).*

Solution. The statement is true. We will again use Riemann's condition for integrability. Let us recall from construction of the Cantor set that $\Delta = \cap_{n=1}^{\infty}\Delta_n$, where Δ_n is the set obtained at the nth stage of the construction. In particular, Δ_n is the union of 2^n chosen intervals each of length 3^{-n}. Let P_n denote the partition on $[0,1]$ generated by the intervals chosen at the nth step of the iteration. Then

$$M_i = 1, m_i = 0 \text{ if } I = [x_{i-1}, x_i] \text{ is a chosen interval,}$$
$$M_i = m_i = 0 \text{ if } I \text{ is not chosen.}$$

This implies that

$$U(\chi_\Delta, P_n) - L(\chi_\Delta, P_n) = \sum_i (M_i - m_i)\Delta x_i = (1)(2^n)(3^{-n}) = \left(\frac{2}{3}\right)^n \to 0 \text{ as } n \to \infty.$$

□

(d) $\bigcap_\alpha \{\mathcal{R}_\alpha[a,b] : \alpha \textit{ increasing}\} = \mathcal{C}[a,b]$.

Solution. The statement is true. We have already seen in class that $\mathcal{C}[a,b] \subseteq \mathcal{R}_\alpha[a,b]$ for any increasing α. Conversely, suppose that $f \in \mathcal{R}_\alpha[a,b]$ for every increasing α. We aim to show that f in continuous on $[a,b]$ and argue by contradiction. Assume if possible that f is discontinuous at the point c (say from the right without loss of generality), and define α to be the increasing function $\chi_{(c,b]}$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ denote any partition of $[a,b]$ in which $x_{i-1} = c$ for some i. (Without loss of generality one can always ensure this after passing to a refinement - why?)

$$U(f,P) - L(f,P) \geq (M_i - m_i)\Delta\alpha_i \geq |f(c+) - f(c)| > 0.$$

Thus Riemann's condition does not hold. □

(e) *If f is a monotone function and α is both continuous and non-decreasing, then $f \in \mathcal{R}_\alpha[a,b]$.*

Solution. The statement is true. Let us fix $\epsilon > 0$ and aim to find a partition P of $[a,b]$ for which $U_\alpha(f,P) - L_\alpha(f,P) < \epsilon$. Since α is continuous on $[a,b]$, it is uniformly continuous; hence there exists $\delta > 0$ such that $\alpha(x) - \alpha(y) < \epsilon$ for all $a \leq y < x \leq b$, $0 < x - y < \delta$. Choosing $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ to be a partition of $[a,b]$ into subintervals of length $< \delta$, and recalling that f is monotone, we find that

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=0}^{n} |f(x_i) - f(x_{i-1})|\Delta\alpha_i \\ &< \epsilon \sum_{i=0}^{n} |f(x_i) - f(x_{i-1})| = \epsilon|f(b) - f(a)|, \end{aligned}$$

where the last step uses the monotonicity of f. Riemann's condition on integrability proves the statement. □

(f) *There exists a non-decreasing function $\alpha : [a,b] \to \mathbb{R}$ and a function $f \in \mathcal{R}_\alpha[a,b]$ such that f and α share a common-sided discontinuity.*

Solution. The statement is false. If a nondecreasing $\alpha : [a,b] \to \mathbb{R}$ and a function $f : [a,b] \to \mathbb{R}$ share a common-sided discontinuity, then $f \notin \mathcal{R}_\alpha[a,b]$.
Suppose that c is a point of common-sided discontinuity, say from the right. Choose a partition P of $[a,b]$ in which c is one of the partitioning points. Then

$$U(f,P) - L(f,P) \geq |f(c+) - f(c)||\alpha(c+) - \alpha(c)|,$$

which violates Riemann's condition for integrability. □

(g) *If $f \in \mathcal{R}_\alpha[a,b]$ with $m \leq f \leq M$ and if φ is continuous on $[m,M]$, then $\varphi \circ f \in \mathcal{R}_\alpha[a,b]$.*

Solution. The statement is true.
Fix $\epsilon > 0$. Since φ is uniformly continuous on $[m,M]$, let us choose $\delta > 0$ such that $|\varphi(u) - \varphi(v)| < \frac{\epsilon}{2(\alpha(b)-\alpha(a))}$ whenever $|u - v| < \delta$. Since f is assumed to be Riemann-Stieltjes integrable on on $[a,b]$, we know from Riemann's condition that there exists a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$ such that

$$U_\alpha(f,P) - L_\alpha(f,P) = \sum_{i=0}^{n} (M_i - m_i)\Delta\alpha_i < \frac{\delta\epsilon}{4||\varphi \circ f||_\infty}, \tag{1}$$

$$M_i(f) = \sup_{x \in I_i} f(x),\ m_i(f) = \inf_{x \in I_i} f(x),\ I_i = [x_{i-1}, x_i].$$

We will decompose the set of indices $i \in \{0, 1, \cdots, n-1\}$ into two disjoint subsets $\mathcal{A}$ and $\mathcal{B}$: $i \in \mathcal{A}$ if $M_i - m_i \leq \delta$ and $i \in \mathcal{B}$ otherwise. Then it follows from (1) above that

$$\delta \sum_{i \in \mathcal{B}} \Delta\alpha_i \leq \sum_{i \in \mathcal{B}} (M_i - m_i)\Delta\alpha_i \leq \frac{\delta\epsilon}{4||\varphi \circ f||_\infty}, \quad \text{i.e.,} \ \sum_{i \in \mathcal{B}} \Delta\alpha_i < \frac{\epsilon}{4||\varphi \circ f||_\infty}.$$

Now, for the same partition P as above we can estimate

$$\begin{aligned}
&U_\alpha(\varphi \circ f, P) - L_\alpha(\varphi \circ f, P) \\
&\qquad = \Big[\sum_{i \in \mathcal{A}} + \sum_{i \in \mathcal{B}}\Big] \big(M_i(\varphi \circ f) - m_i(\varphi \circ f)\big)\Delta\alpha_i \\
&\qquad \leq \frac{\epsilon}{2(\alpha(b) - \alpha(a))} \sum_{i \in \mathcal{A}} \Delta\alpha_i + 2||\varphi \circ f||_\infty \sum_{i \in \mathcal{B}} \Delta\alpha_i \\
&\qquad \leq \frac{\epsilon}{2(\alpha(b) - \alpha(a))}(\alpha(b) - \alpha(a)) + 2||\varphi \circ f||_\infty \frac{\epsilon}{4||\varphi \circ f||_\infty} \\
&\qquad \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

verifying Riemann's condition. □