1. Given a nonconstant non-decreasing function $\alpha : [a, b] \to \mathbb{R}$, let $\mathcal{R}_\alpha[a, b]$ denote the collection of all bounded functions on $[a, b]$ which are Riemann-Stieltjes integrable with respect to $\alpha$. Is $\mathcal{R}_\alpha[a, b]$ a vector space, a lattice, an algebra?

Solution. They are all a vector space, a lattice, an algebra. (See also Theorem 6.12-6.13 in the textbook.) To show that $f, g \in \mathcal{R}_\alpha[a, b]$ implies $f + g, cf, |f| \in \mathcal{R}_\alpha[a, b]$, we use (verify it!)

$$U_\alpha(f + g, P) - L_\alpha(f + g, P) = (U_\alpha(f, P) - L_\alpha(f, P)) + (U_\alpha(g, P) - L_\alpha(g, P))$$

$$U_\alpha(cf, P) - L_\alpha(cf, P) = cU_\alpha(f, P) - cL_\alpha(g, P)$$

$$U_\alpha(|f|, P) - L_\alpha(|f|, P) \leq U_\alpha(f, P) - L_\alpha(g, P)$$

To show that $fg \in \mathcal{R}_\alpha[a, b]$. First we prove in the special case that $f, g$ are non-negative then general case follows by decomposing $f = f^+ - f^-, g = g^+ - g^-$ and we know that $f^+, f^-, g^+, g^- \in \mathcal{R}_\alpha[a, b]$ since it is a lattice.

Now assume that $f, g$ are non-negative, we use

$$|U_\alpha(fg, P) - L_\alpha(fg, P)| \leq |U_\alpha(f, P)U_\alpha(g, P) - L_\alpha(f, P)L_\alpha(g, P)|$$

$$\leq |U_\alpha(f, P)||U_\alpha(g, P) - L_\alpha(g, P)| + |L_\alpha(g, P)||U_\alpha(f, P) - L_\alpha(f, P)|$$

Here $|U_\alpha(f, P)|, |L_\alpha(g, P)|$ are bounded by an absolute constant.

2. This problem focuses on computing the Riemann-Stieltjes integral for specific choices of integrators.

(a) Let $x_0 = a < x_1 < x_2 < \cdots < x_n = b$ be a fixed collection of points in $[a, b]$, and let $\alpha$ be an increasing step function on $[a, b]$ that is constant on each of the open intervals $(x_{i-1}, x_i)$ and has jumps of size $\alpha_i = \alpha(x_i+) - \alpha(x_i-)$ at each of the points $x_i$. For $i = 0$ and $n$, we make the obvious adjustments

$$\alpha_0 = \alpha(a+) - \alpha(a-), \quad \alpha_n = \alpha(b) - \alpha(b-).$$

If $f \in B[a, b]$ is continuous at each of the points $x_i$, show that $f \in \mathcal{R}_\alpha[a, b]$ and

$$\int_a^b f \, d\alpha = \sum_{i=0}^n f(x_i)\alpha_i.$$

Solution. Intuitively, the point $x_i$ will contribute the value of the integral if $\alpha$ jumps at $x_i$ which $\alpha_i$ measures how large does $\alpha$ jump. . Given $\epsilon > 0$, for each $x_i$ there is a $\delta_i$ such that $|x - x_i| < \delta \implies |f(x) - f(x_i)| < \epsilon$. Choose $\delta = \min\{\delta_1, \ldots, \delta_n\}$ then $|f(x) - f(x_i)| < \epsilon$. (or one can argue that $f$ is uniformly continuous). Also assume $\delta < \frac{1}{2}\min_{1 \leq i \leq n} |x_i - x_{i+1}|$.

Refine the partition $P$ as follows:

- For each $x_0, x_1, \ldots, x_{n-1}$ choose a point $y_i \in (x_i, x_{i+1})$.
- For each $x_1, x_2, \ldots, x_n$ choose a point $z_i \in (x_i - \delta, x_i)$. Then we have a new partition $P^*$

$$\{x_0, y_0, z_1, x_1, y_1, z_2, x_2, y_2, \ldots, z_{n-1}, x_{n-1}, y_{n-1}, z_n, x_n\}$$
Now using definition of $\delta$ and the fact that $\alpha$ is constant along $(x_i, x_{i+1})$

$$L_\alpha(f, P^*) \geq \sum_{i=1}^{n} (f(x_i) - \epsilon)\alpha_i$$

$$U_\alpha(f, P^*) < \sum_{i=1}^{n} (f(x_i) + \epsilon)\alpha_i$$

We have

$$\sum_{i=1}^{n} (f(x_i) - \epsilon)\alpha_i < L_\alpha(f, P^*) \leq \int_{a}^{b} f \, d\alpha < U_\alpha(f, P^*) < \sum_{i=1}^{n} (f(x_i) + \epsilon)\alpha_i$$

Since $\epsilon > 0$ is arbitrary (small), let $\epsilon \to 0$, we are done.

(b) If $f$ is continuous on $[1, n]$, compute $\int_{1}^{n} f(x)d[x]$, where $[x]$ is the greatest integer in $x$.

What is the value of $\int_{1}^{t} f(x)d[x]$ if $t$ is not an integer?

Solution. Apply part a) with $x_0 = 1, x_1 = 2, \ldots, x_{n-1} = n$ since $\alpha_0 = 0, \alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 1$, we have

$$\int_{1}^{n} f[x] = \sum_{i=0}^{n-1} f(n)\alpha_i = \sum_{k=2}^{n} f(k)$$

i.e. on $[1, N], [x]$ jumps at $2, 3, \ldots, N$. In the same way,

$$\int_{1}^{t} f[d[x] = \sum_{k=2}^{[t]} f(k)$$

In we want the sum from 1 to $n$ we could take e.g.

$$\int_{1}^{t} f[d[x] = \sum_{k=1}^{[t]} f(k)$$

where 0.99 could be replaced by any numbers in $[0, 1)$.  \[ \square \]

3. Determine, with adequate justification, whether each of the following statements is true or false.

(a) An equicontinuous, pointwise bounded subset of $C[a, b]$ is compact.

Solution. The statement is easily seen to be false; the class of constant functions $\{\frac{1}{n} : n \geq 1\}$ provides a counterexample. The collection is equicontinuous but not closed (the constant function zero is a limit point outside the set), hence not compact.

Remark: Note however that an equicontinuous, pointwise bounded and closed subset of $C(X)$ is in fact compact. Just follow the proof of the Arzela-Ascoli theorem.  \[ \square \]

(b) The function $\chi_\mathbb{Q}$ is Riemann integrable on $[0, 1]$.

Solution. The function $\chi_\mathbb{Q}$ is not Riemann integrable on $[0, 1]$ or on any interval $[a, b]$ for that matter. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, it follows that $U(\chi_\mathbb{Q}, P) = 1$ and $L(\chi_\mathbb{Q}) = 0$ for any partition $P$ of $[a, b]$. By Riemann’s condition on integrability, $\chi_\mathbb{Q}$ fails to be integrable.  \[ \square \]
(c) The function \( \chi_\Delta \) is Riemann integrable on \([0, 1] \), where \( \Delta \) denotes the Cantor middle-third set. (We have already run into this set in Homework 2, Problem 5).

**Solution.** The statement is true. We will again use Riemann’s condition for integrability. Let us recall from construction of the Cantor set that \( \Delta = \cap_{n=1}^\infty \Delta_n \), where \( \Delta_n \) is the set obtained at the \( n \)th stage of the construction. In particular, \( \Delta_n \) is the union of \( 2^n \) chosen intervals each of length \( 3^{-n} \). Let \( P_n \) denote the partition on \([0, 1]\) generated by the intervals chosen at the \( n \)th step of the iteration. Then
\[
M_i = 1, m_i = 0 \text{ if } I = [x_{i-1}, x_i] \text{ is a chosen interval,}
M_i = m_i = 0 \text{ if } I \text{ is not chosen.}
\]
This implies that
\[
U(\chi_\Delta, P_n) - L(\chi_\Delta, P_n) = \sum_i (M_i - m_i) \Delta x_i = (1)(2^n)(3^{-n}) = \left(\frac{2}{3}\right)^n \to 0 \text{ as } n \to \infty.
\]
\( \square \)

(d) \( \bigcap_\alpha \{R_\alpha[a, b] : \alpha \text{ increasing} \} = C[a, b] \).

**Solution.** The statement is true. We have already seen in class that \( C[a, b] \subseteq R_\alpha[a, b] \) for any increasing \( \alpha \). Conversely, suppose that \( f \in R_\alpha[a, b] \) for every increasing \( \alpha \). We aim to show that \( f \) is continuous on \([a, b] \) and argue by contradiction. Assume if possible that \( f \) is discontinuous at the point \( c \) (say from the right without loss of generality), and define \( \alpha \) to be the increasing function \( \chi_{(c,b)} \). Let \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \) denote any partition of \([a, b] \) in which \( x_{i-1} = c \) for some \( i \). (Without loss of generality one can always ensure this after passing to a refinement - why?)
\[
U(f, P) - L(f, P) \geq (M_i - m_i) \Delta x_i = |f(c^+) - f(c)| > 0.
\]
Thus Riemann’s condition does not hold. \( \square \)

(e) If \( f \) is a monotone function and \( \alpha \) is both continuous and non-decreasing, then \( f \in R_\alpha[a, b] \).

**Solution.** The statement is true. Let us fix \( \epsilon > 0 \) and aim to find a partition \( P \) of \([a, b] \) for which \( U_\alpha(f, P) - L_\alpha(f, P) < \epsilon \). Since \( \alpha \) is continuous on \([a, b] \), it is uniformly continuous; hence there exists \( \delta > 0 \) such that \( \alpha(x) - \alpha(y) < \epsilon \) for all \( a \leq y < x \leq b, 0 < x - y < \delta \). Choosing \( P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\} \) to be a partition of \([a, b] \) into subintervals of length < \( \delta \), and recalling that \( f \) is monotone, we find that
\[
U(f, P) - L(f, P) = \sum_{i=0}^n |f(x_i) - f(x_{i-1})| \Delta x_i < \epsilon \sum_{i=0}^n |f(x_i) - f(x_{i-1})| = \epsilon |f(b) - f(a)|,
\]
where the last step uses the monotonicity of \( f \). Riemann’s condition on integrability proves the statement. \( \square \)

(f) There exists a non-decreasing function \( \alpha : [a, b] \to \mathbb{R} \) and a function \( f \in R_\alpha[a, b] \) such that \( f \) and \( \alpha \) share a common-sided discontinuity.
Solution. The statement is false. If a nondecreasing \( \alpha : [a,b] \to \mathbb{R} \) and a function \( f : [a,b] \to \mathbb{R} \) share a common-sided discontinuity, then \( f \notin R_\alpha[a,b] \).

Suppose that \( c \) is a point of common-sided discontinuity, say from the right. Choose a partition \( P \) of \([a,b]\) in which \( c \) is one of the partitioning points. Then

\[
U(f, P) - L(f, P) \geq |f(c+) - f(c)||\alpha(c+) - \alpha(c)|,
\]

which violates Riemann’s condition for integrability.

\(\square\)

(g) If \( f \in R_\alpha[a,b] \) with \( m \leq f \leq M \) and if \( \varphi \) is continuous on \([m,M]\), then \( \varphi \circ f \in R_\alpha[a,b] \).

Solution. The statement is true.

Fix \( \epsilon > 0 \). Since \( \varphi \) is uniformly continuous on \([m,M]\), let us choose \( \delta > 0 \) such that

\[
|\varphi(u) - \varphi(v)| < \frac{\epsilon}{2(\alpha(b) - \alpha(a))}
\]

whenever \( |u - v| < \delta \). Since \( f \) is assumed to be Riemann-Stieltjes integrable on \([a,b]\), we know from Riemann’s condition that there exists a partition \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \) of \([a,b]\) such that

\[
U_\alpha(f, P) - L_\alpha(f, P) = \sum_{i=0}^{n} (M_i - m_i) \Delta \alpha_i < \frac{\delta \epsilon}{4||\varphi \circ f||_\infty},
\]

\[
M_i(f) = \sup_{x \in I_i} f(x), \ m_i(f) = \inf_{x \in I_i} f(x), \ I_i = [x_{i-1}, x_i].
\]

We will decompose the set of indices \( i \in \{0,1, \cdots, n-1\} \) into two disjoint subsets \( \mathcal{A} \) and \( \mathcal{B} \): \( i \in \mathcal{A} \) if \( M_i - m_i \leq \delta \) and \( i \in \mathcal{B} \) otherwise. Then it follows from (1) above that

\[
\delta \sum_{i \in \mathcal{B}} \Delta \alpha_i \leq \sum_{i \in \mathcal{B}} (M_i - m_i) \Delta \alpha_i \leq \frac{\delta \epsilon}{4||\varphi \circ f||_\infty}, \quad \text{i.e.,} \quad \sum_{i \in \mathcal{B}} \Delta \alpha_i < \frac{\epsilon}{4||\varphi \circ f||_\infty}.
\]

Now, for the same partition \( P \) as above we can estimate

\[
U_\alpha(\varphi \circ f, P) - L_\alpha(\varphi \circ f, P)
= \left[ \sum_{i \in \mathcal{A}} + \sum_{i \in \mathcal{B}} \right] (M_i(\varphi \circ f) - m_i(\varphi \circ f)) \Delta \alpha_i
\leq \frac{\epsilon}{2(\alpha(b) - \alpha(a))} \sum_{i \in \mathcal{B}} \Delta \alpha_i + 2||\varphi \circ f||_\infty \sum_{i \in \mathcal{B}} \Delta \alpha_i
\leq \frac{\epsilon}{2(\alpha(b) - \alpha(a))} (\alpha(b) - \alpha(a)) + 2||\varphi \circ f||_\infty \frac{\epsilon}{4||\varphi \circ f||_\infty}
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]

verifying Riemann’s condition. \(\square\)