Homework 5 Solutions- Math 321, Spring 2015

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1. *The classical Weierstrass approximation theorem says that the class of polynomials is dense in $\mathcal{C}[a,b]$. The Stone-Weierstrass theorem, on the other hand, provides a necessary and sufficient condition for a subalgebra of $\mathcal{C}(X)$ to be dense if X is compact, but does not seem to furnish a concrete dense class of functions akin to the polynomials. While we do not have anything so convenient as polynomials at our disposal for a general compact X, we do have a familiar collection of functions to work with that for many purposes serves as an adequate replacement. Here it is:*

   Given a metric space (X,d) and a constant $0 \le K < \infty$, let $\mathcal{L}_K(X)$ denote the collection of all real-valued Lipschitz functions on A with Lipschitz constant at most K; in other words, $f : X \to \mathbb{R}$ is in $\mathcal{L}_K(X)$ if

$$|f(x) - f(y)| \le K d(x,y) \qquad \text{for all } x, y \in X. \tag{1}$$

   We write $\mathcal{L}(X)$ to denote the set of functions that are in $\mathcal{L}_K(X)$ for some finite K; i.e., $\mathcal{L}(X) = \bigcup_{K=1}^{\infty} \mathcal{L}_K(X)$.

   (a) *Clearly all constant functions are Lipschitz. Show that $\mathcal{L}(X)$ contains non-constant functions as well.*

   Solution. For any $x_0 \in X$, define the function $f_{x_0}(x) = d(x, x_0)$. Since $f_{x_0}(x) = 0$ for $x = x_0$, and nonzero otherwise, it follows that f_{x_0} is nonconstant. On the other hand,

$$|f_{x_0}(x) - f_{x_0}(y)| = |d(x,x_0) - d(y,x_0)| \le d(x,y)$$

   by the triangle inequality, so f_{x_0} is Lipschitz with Lipschitz constant at most 1. □

   (b) *Show that for an arbitrary metric space (X,d), $\mathcal{L}(X)$ is a subspace of $\mathcal{C}(X)$.*

   Solution. We first verify that any Lipschitz function is continuous, i.e., $\mathcal{L}(X) \subseteq \mathcal{C}(X)$. Fix $f \in \mathcal{L}_K(X)$ and $\epsilon > 0$. Setting $\delta = \epsilon/K$ and applying the Lipschitz condition (1) we find that

$$d(x,y) < \delta \quad \Longrightarrow \quad |f(x) - f(y)| \le K d(x,y) < K\delta = \epsilon, \tag{2}$$

   which is the $\epsilon - \delta$ definition of continuity.
   Next we need to prove that $\mathcal{L}(X)$ is a subspace of $\mathcal{C}(X)$. Since $\mathcal{C}(X)$ is itself a vector space, it suffices to show that $\mathcal{L}(X)$ is closed under vector addition and scalar multiplication. If $f \in \mathcal{L}_K(X)$ and $c \in \mathbb{R}$, multiplying both sides of (1) by $|c|$ shows that $cf \in \mathcal{L}_{cK}(X)$. On the other hand, if $f \in \mathcal{L}_K(X)$ and $g \in \mathcal{L}_M(X)$, then the triangle inequality combined with (1) gives $f + g \in \mathcal{L}_{K+M}(X)$. □

   (c) *If X is compact, show that $\mathcal{L}(X)$ is in fact a subalgebra of $\mathcal{C}(X)$.*

   Solution. We have already established in part (b) that $\mathcal{L}(X)$ is subspace of $\mathcal{C}(X)$. Since $\mathcal{C}(X)$ is itself an algebra, it suffices to show that $\mathcal{L}(X)$ is closed under vector multiplication. Suppose that $f, g \in \mathcal{L}(X)$. Then there exists $K > 0$ such that $f, g \in \mathcal{L}_K(X)$.

Since f and g are continuous on a compact set, they are both bounded. Hence,

$$\begin{aligned}|f(x)g(x) - f(y)g(y)| &\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\ &\leq K(||f||_\infty + ||g||_\infty)d(x, y),\end{aligned}$$

showing that $fg \in \mathcal{L}_M(X)$ with $M = K(||f||_\infty + ||g||_\infty) < \infty$. □

(d) *Given an arbitrary metric space X, show that $\mathcal{L}(X)$ separates points of X and vanishes at no point of X. If X is compact, deduce from this that $\mathcal{L}(X)$ is dense in $\mathcal{C}(X)$.*

Solution. The functions $f_{x_0} \in \mathcal{L}(X)$ separate points and the constant function 1 vanishes at no point. Since we have already shown that $\mathcal{L}(X)$ is a subalgebra, its density in $\mathcal{C}(X)$ (for X compact) is a consequence of the Stone-Weierstrass theorem. □

2. *Show that any compact subset of a metric space is separable, and deduce from it that a countable union of compact sets is separable as well.*

Solution. Let K be a compact subset of a metric space (X, d) then for each n, $\{B_{\frac{1}{n}}(x)\}_{x\in K}$ is an open cover for K hence it has a finite subcover $\{B_{\frac{1}{n}}(x_i^{(n)})\}_{i=1}^{N_n}$. For each n let $E_n = \{x_1^{(n)}, \cdots, x_{N_n}^{(n)}\}$. Define $E = \bigcup_{n=1}^\infty E_n$. Hence E is countable. We only need to show that E is dense in K then we are done.

To see this, let $x \in K$ and $\epsilon > 0$, choose n such that $n > \frac{1}{\epsilon}$ then for a $p \in E_n \subseteq E$, we have $d(p, x) < \frac{1}{n} < \epsilon$. So E is dense in K.

Now if we have a countable union of compact sets: $E = \bigcup_{n=1}^\infty E_n$ then each compact set E_i has a countable dense subset D_i then it is not hard to see that $= \bigcup_{n=1}^\infty D_n$ is countable and dense in E.

□

3. *One of the applications of the classical Weierstrass theorem was to prove that $\mathcal{C}[a, b]$ is separable. Likewise, the Stone-Weierstrass theorem can be used to show that $\mathcal{C}(X)$ is separable where X is a compact metric space. This exercise aims to give a proof of this statement.*

(a) *Recall the class $\mathcal{L}(X)$ of Lipschitz functions introduced in Problem 1. Use Problem 2 to argue that $\mathcal{L}(X)$ is separable if X is compact. (*Hint: *You may want to look at bounded subsets of $\mathcal{L}_K(X)$.)*

Solution. Let us define

$$\mathcal{E}_K = \{f \in \mathcal{L}_K(X) : ||f||_\infty \leq K\}. \tag{3}$$

We will prove in Lemma 0.1 below that each $\mathcal{E}_K$ is compact in $\mathcal{C}(X)$ and hence has a countable dense subset by Problem 2. On the other hand, $\mathcal{L}(X) = \bigcup_{K=1}^\infty \mathcal{E}_K$. This is because any $f \in \mathcal{L}(X)$ lies in $\mathcal{L}_M(X)$ for some $M > 0$, and is also bounded in sup norm by compactness of X as explained above. Thus $f \in \mathcal{E}_K$, for any $K > M + ||f||_\infty$. The second conclusion of Problem 2 then yields that $\mathcal{L}(X)$ is a countable union of compact sets, hence separable. □

Lemma 0.1. *Each set $\mathcal{E}_K$ defined in* (3) *is compact in $\mathcal{C}(X)$.*

Proof of Lemma. We argue that each $\mathcal{E}_K$ is uniformly bounded, closed and equicontinuous, hence compact by the Arzela-Ascoli theorem. Clearly every function in $\mathcal{E}_K$ is uniformly bounded by the constant K. To see that $\mathcal{E}_K$ is closed, we simply observe that $\mathcal{E}_K$ is the intersection of $\mathcal{L}_K(X)$ with the closed ball in $\mathcal{C}(X)$ centred at 0 with radius K. Closure of the set $\mathcal{E}_K$ would follow if we are able to show that $\mathcal{L}_K(X)$ is closed. But (1), which is the defining inequality of the class $\mathcal{L}_K(X)$, is preserved under limits. This establishes that $\mathcal{L}_K(X)$ is closed, as desired. To deduce that $\mathcal{E}_K$ is equicontinuous, we note that the value of the continuity parameter $\delta(=\epsilon/K)$ that we obtained in the proof of (2) depends only on K and ϵ, but is independent of functions f in $\mathcal{L}_K(X)$. $\square$

(b) *Conclude from part (a) that $\mathcal{C}(X)$ is separable.*

Proof. Since $\mathcal{L}(X)$ is dense in $\mathcal{C}(X)$ (by Problem 1(d)) and separable by Problem 3(a), it follows that $\mathcal{C}(X)$ admits a countable dense set $\mathcal{L}(X)$, and is therefore separable. $\square$

4. *You have seen examples showing that pointwise convergence does not imply uniform convergence. However, pointwise convergence combined with equicontinuity does imply convergence in this stronger sense. Prove this. More precisely, show that if X is a compact metric space, then any equicontinuous sequence of functions in $\mathcal{C}(X)$ that is pointwise convergent is in fact uniformly convergent.*

Solution. Let $\{f_n\}$ be equicontinuous and converges pointwise. Let $\epsilon > 0$ then there exists $\delta > 0$ such that for all n and $x, y \in X, d(x,y) < \delta \implies |f_n(x) - f_n(y)| < \frac{\epsilon}{3}$. Now $\{B_\delta(x)\}_{x\in X}$ is an open cover for X and hence has a finite subcover $B_\delta(x_1), \cdots, B_\delta(x_M)$ i.e. for all $x \in X$ there is a k such that $d(x, x_k) < \delta$. Now f converges pointwise at $x_1, \cdots, x_M$ hence $\exists N_1, ..., N_M, \forall 1 \leq k \leq m; n, m \geq N_k \implies |f_n(x_k) - f_m(x_k)| < \frac{\epsilon}{3}$. Since there are finitely many of N_i let $N = \max\{N_1, \cdots, N_k\}$ we have

$$n, m > N \implies |f_n(x_i) - f_m(x_i)| < \frac{\epsilon}{3} \ \forall i.$$

Now let $x \in X$ choose an x_k such that $d(x, x_k) < \delta$. Now if $n, m > N$ then for all $x \in X$,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_k)| + |f_n(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(x)| < \epsilon.$$

This is true uniformly in x (i.e. the same N works for all x.) Hence $\{f_n\}$ is uniform cauchy. So it is uniformly convergent.
Note Using cauchy sequence is a convenient way to show convergence. If you want do it directly i.e. trying to bound $|f(x) - f_n(x)|$ where f is the limit function. One end up with bounding $|f(x) - f(y)|, d(x, y) < \delta$ (This does not implies $\|f_n - f\|_\infty \to 0$ in $C(X)$ as it is only pointwise convergence). It is not obvious that f is continuous.
However the statement is actually true, we have $|f_n(x) - f_n(y)| < \epsilon$ for $d(x, y) < \delta$. One may take limit (we already know that limit exists)

$$\lim_{n\to\infty} |f_n(x) - f_n(y)| = |f(x) - f(y)| \leq \epsilon.$$

However, to take limit one has to do it carefully. We can take limit because it is equicontinuous and the same δ works for all n. In general (e.g. f_n are merely uniformly continuous for all n) then this argument is not valid as this δ may not work for other n. So one has to be careful and precise in his/her arguments. $\square$

5. *Let $\{f_n\}$ be an equicontinuous sequence in $\mathcal{C}[a,b]$ such that $\{f_n\}$ converges pointwise at every rational in $[a,b]$. Prove that $\{f_n\}$ converges uniformly on $[a,b]$.*

Solution. First we show that $\{f_n\}$ converges pointwise on $[a,b]$ by showing that $\{f_n(x)\}$ is cauchy for each $x \in [a,b]$.Let $x \in X$. Let $\epsilon > 0$ then since $\{f_n\}$ is equicontinuous, there is $\delta > 0$ such that for all $x, y \in [a,b], |x-y| < \delta \implies |f_n(x) - f_n(y)| < \frac{\epsilon}{3}$. Since $\mathbb{Q}$ in dense in $\mathbb{R}$ we can find $q \in \mathbb{Q} \bigcap [a,b]$ such that $|x-q| < \delta$ and hence $|f_n(x) - f_n(q)| < \frac{\epsilon}{3}$ for all n.

Now since f_n converges pointwise on rational, there is N such that $n, m > N \implies |f_n(q) - f_m(q)| < \frac{\epsilon}{3}$. Hence if $n, m > N$ then

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(q)| + |f_n(q) - f_m(q)| + |f_m(q) - f_m(x)| < \epsilon.$$

Hence f_n converges (pointwise) at any point $x \in [a,b]$. Using results form Q4 we have uniform convergence. □