1. Find the limit of the function \( f(z) = (z/\bar{z})^2 \), if it exists, as \( z \) tends to zero. If you think the limit does not exist, explain your reasoning for this conclusion.

Solution. If \( z \to 0 \) along the line \( y = mx \), then \( z = x + imx = x(1 + im) \), \( \bar{z} = x - imx = x(1 - im) \), hence

\[
f(z) = f(x + imx) = \left( \frac{x(1 + im)}{x(1 - im)} \right)^2 = \frac{(1 - m^2) + 2im}{(1 - m^2) - 2im}.
\]

This last quantity depends on \( m \). For example, it is 1 if \( m = 0 \), i.e., when \( z \to 0 \) along the \( x \)-axis. The value is \((-3 + 4im)/(-3 - 4im) \neq 1 \) if \( m = 2 \). Since the limiting value of \( f \) depends on the angle of approach, \( \lim_{z \to 0} f(z) \) does not exist.

Caution! This problem is not about the differentiability of the function \( f \), so please do not use the dependence on \( \bar{z} \) to deduce that the limit does not exist.

\[\square\]

2. Describe geometrically the collection of points \( z \) satisfying the equation \( |z-1| = |z+i| \). Sketch this set of points in the complex plane.

Solution. Recall that \( |z-z_0| \) is the distance of the point \( z \) from \( z_0 \). Thus the equation \( |z-1| = |z+i| \) represents all points \( z \) which are equidistant from 1 and \(-i\). Such points lie on the perpendicular bisector of the line segment joining 1 and \(-i\). Thus the collection of \( z \) satisfying the equation is the infinite line passing through the point \((0,0)\) with slope \(-1\).

An alternative strategy: You could also try to simplify the equation \((x-1)^2 + y^2 = x^2 + (y+1)^2\).

\[\square\]

3. Express the complex number \((-1 + i)^7\) in the form \( a + ib \).

Solution. We write \(-1 + i\) in polar form: \(-1 + i = \sqrt{2}e^{\frac{3\pi}{4}}\). Therefore

\[
(-1 + i)^7 = (\sqrt{2})^7 e^{\frac{21\pi i}{4}} = 8\sqrt{2}e^{\frac{5\pi i + \pi i}{4}} = -8\sqrt{2}\frac{1+i}{\sqrt{2}} = -8(1+i).
\]

\[\square\]

4. Decide whether the set \( \{z : 0 \leq \arg(z) \leq \frac{\pi}{4}\} \) is bounded. Give reasons for your answer.

Solution. The set \( \{z : 0 \leq \arg(z) \leq \frac{\pi}{4}\} \) is the infinite triangular region in the first quadrant bounded by the lines \( y = 0 \) and \( y = x \). This region cannot be contained within a ball of any finite radius, and is hence unbounded.

\[\square\]

5. Describe the domain of definition of the function \( f(z) = z/(z + \bar{z}) \).
Solution. The functions $z$ and $z + \bar{z}$ are well-defined on the whole complex plane. Their ratio is defined whenever the denominator is nonzero. But $z + \bar{z} = 0$ if and only if $z = -\bar{z}$, i.e., $x + iy = -x + iy$ or $x = \text{Re}(z) = 0$. Therefore the domain of definition of $f$ is \( \{ z \in \mathbb{C} : \text{Re}(z) \neq 0 \} \). \qed

6. Find and sketch the images of the hyperbolas

\[ x^2 - y^2 = -1 \quad \text{and} \quad xy = -2 \]

under the transformation $w = z^2 = (x + iy)^2$.

Solution. Observe that

\[ z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy = u + iv, \]

so the set of $z = x + iy$ with $x^2 - y^2 = -1$ maps to $u + iv = -1 + 2ixy$, which is contained in the vertical line $u = -1$ in the $(u,v)$ plane. Conversely, given any point of the form $-1 + ik$ on this line, there exist values of $(x,y)$ satisfying

\[ x^2 - y^2 = -1 \quad \text{and} \quad 2xy = k. \]

This can be seen by substituting $y = k/(2x)$ from the second equation into the first, obtaining a quadratic equation in $x^2$, namely

\[ x^2 - \left(\frac{k}{2x}\right)^2 = -1, \quad \text{or} \quad 4x^4 + 4x^2 - k^2 = 0. \]

The last equation has a non-negative solution $x^2 = (-4 + \sqrt{16 + 16k^2})/8$. Thus the image of the hyperbola $x^2 - y^2 = -1$ under the squaring map is the entire line $u = -1$.

Similarly, the set of $z = x + iy$ with $xy = -2$ maps to $p + iq = x^2 - y^2 - 4i$ which is a point on the horizontal line $q = -4$ in the $(p,q)$ plane. As above, one can show that every point $k - 4i$ on this line is in fact the image of some $(x, y)$ on the hyperbola $xy = -2$. To see this, we need to show that there exist $(x, y)$ that satisfy the two equations

\[ xy = -2 \quad x^2 - y^2 = k. \]

Upon eliminating $y$, this reduces to solving the equation $x^4 - kx^2 - 4 = 0$, which admits a real solution in $x$. Thus the image of the hyperbola in the entire line $q = -4$. \qed

7. Show that the function $f(z) = x^2 + iy^2$ is differentiable at the origin but analytic nowhere.

Solution. Set $u(x,y) = x^2$ and $v(x,y) = y^2$. Then $u_x = 2x$, $u_y = v_x = 0$ and $v_y = 2y$. Thus there is no open set on which the Cauchy-Riemann equations hold. Therefore $f$ is not analytic on any open set in the complex plane.

We will now show that $f$ is differentiable at the origin and that $f'(0) = 0$.

\[
\left| \lim_{(x,y) \to (0,0)} \frac{f(x+iy) - 0}{x+iy} \right| = \lim_{(x,y) \to (0,0)} \frac{|x^2 + iy^2|}{|x + iy|} = \lim_{(x,y) \to (0,0)} \frac{\sqrt{x^4 + y^2}}{\sqrt{x^2 + y^2}}. \]
Since the expression above is symmetric in $x$ and $y$, we may assume without loss of generality that $|x| \geq |y|$. With this assumption, we see that
\[
\frac{\sqrt{x^4 + y^2}}{\sqrt{x^2 + y^2}} \leq \frac{2x^4}{x^2} = 2x^2 \to 0,
\]
hence the limit exists, and its value is zero. □

8. **Find the harmonic conjugate of the function** $u(x, y) = y^3 - 3x^2y$ **if it exists.** If the answer is yes, determine the analytic function $f$ whose real part is $u$.

**Solution.** If $v$ is the harmonic conjugate of $u$, then by definition $f = u + iv$ is analytic. Therefore $u, v$ must satisfy the Cauchy-Riemann equations
\[
\begin{align*}
u_x &= -6xy = v_y \quad \text{and} \quad u_y = 3y^2 - 3x^2 = -v_x. 
\end{align*}
\]
This implies that $v = -3xy^2 + A(x) = -3xy^2 + x^3 + B(y)$. Thus $v = -3xy^2 + x^3 + C$, where $C$ is an arbitrary constant.

Notice that if $f = u + iv$, then $f(x, 0) = u(x, 0) + iv(x, 0) = i(x^3 + C)$. This suggests the possibility that $f(z) = i(z^3 + C)$, which one can now easily verify:
\[
f(z) = y^3 - 3x^2y + i(-3xy^2 + x^3 + C) = i(z^3 + C).
\]

□

9. **State whether each of the following statements is true or false.** If the statement is true, give a short proof of it. If not, give a counterexample to show that it is false.

(a) **The function** $f(z) = e^z$ **is harmonic.**

**Solution.** True. The function $f$ is entire, i.e., satisfies the Cauchy-Riemann equations. Since Laplace’s equation follows from the Cauchy-Riemann equations, $f$ is harmonic. □

(b) $|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}|2z + 5|$.

**Solution.** True. $|(2\bar{z} + 5)(\sqrt{2} - i)| = |(2\bar{z} + 5)| \times |(\sqrt{2} - i)| = \sqrt{3}|(2\bar{z} + 5)| = \sqrt{3}|2z + 5|$. □

(c) **There exists a complex number** $z_0$ **whose fourth roots** $z_1, z_2, z_3, z_4$ **have the property that**
\[
\arg(z_1) = \frac{\pi}{4}, \quad \arg(z_2) = \frac{\pi}{2}, \quad \arg(z_3) = \frac{2\pi}{3}, \quad \arg(z_4) = \pi.
\]

**Solution.** False. The fourth roots of any complex number are equispaced points on a circle centred at the origin. The argument of each root has to be separated from that of its nearest neighbour by $\pi/2$. This is not the case here. □

(d) **The equation** $(z^2 + z + 1)e^z = 0$ **has exactly two complex roots.**
Solution. True. $e^z = e^a (\cos y + i \sin y)$ has no zero in $\mathbb{C}$, so the equation reduces to finding the roots of the quadratic polynomial $z^2 + z + 1$. By the fundamental theorem of algebra, this polynomial has exactly two complex roots.

(e) If a rational function $R$ has a pole at the point $a$, then the residue of $R$ at $a$ must be a nonzero complex number.

Solution. False. The function $R(z) = 1/z^2$ has a pole at $z = 0$, but its residue at that point is 0.