Midterm Exam II
November 5, 2014

No books. No notes. No calculators. No electronic devices of any kind.

Name ___________________ Student Number ___________________

Problem 1. (3 points)
Find all values of $i^i$.

\[
 i^i = e^{\log(i^i)} = e^{i \log(i)} = e^{\frac{\ln |i|}{2} + i \cdot \arg(i)}
\]

\[
 = e^{i(\pi + \cdot \arg(i))} = e^{-\arg(i)} = e^{-\frac{\pi}{2} + 2\pi k}, k \in \mathbb{Z}
\]

**Surprise:** all values of $i^i$ are positive real!
Problem 2. (3 points)
Carefully sketch the branch cut(s) of the principal branch of the multi-valued function

\[ f(z) = \sqrt[3]{z^3 - 27}. \]

(The principal branch is defined in terms of the principal branch of the logarithm.)

\[ f'(z) = \left(\frac{z^3 - 27}{2}\right)^{\frac{1}{3}} = e^{\frac{1}{3} \log(z^3 - 27)} \]

So the branch cuts of \( f(z) \) are where \( z^3 - 27 = -r \quad (r > 0 \text{ real}) \).

Or \( z^3 = 27 - r \)

For \( r = 0 \) : \( z^3 = 27 \) so \( z = 3e^{2\pi i k/3} \quad k = 0, 1, 2 \)

For \( r = 27 \) : \( z^3 = 0 \) so \( z = 0 \)

For \( r = 54 \) : \( z^3 = -27 \) so \( z = 3e^{2\pi i/3} + 2\pi i k/3 \quad k = 0, 1, 2 \)

etc.
Problem 3. (5 points)
True or false? (No reasons necessary.)
(a) Every branch of the multi-valued function \( f(z) = i^z \) is an entire function.
(b) The principal branch of the function \( f(z) = z^i \) is analytic at \( z = i \).
(c) The function \( f(z) = \sinh(z) \) is periodic, with period \( 2\pi i \).
(d) Let \( \Gamma \) be a circle centered at the origin, traversed once in the counterclockwise direction. Then, for every complex number \( z \) not on \( \Gamma \), we have
\[
\oint_{\Gamma} \frac{w}{w-z} \, dw = \begin{cases} 
2\pi i & \text{if } z \text{ is inside } \Gamma, \\
0 & \text{if } z \text{ is outside } \Gamma.
\end{cases}
\]
(e) For every closed contour \( \Gamma \), which avoids the origin, we have
\[
\oint_{\Gamma} \frac{1+z^{10}}{z^{100}} \, dz = 0.
\]
(a) \( f(z) = i^z = e^{z \log(i)} \). For every value of \( \log(i) \), for example \( \log(i) = \frac{i\pi}{2} \),
this gives an entire function, for example \( e^{z i \pi/2} \).
Another value of \( \log(i) \) is \( \frac{5i\pi}{4} \), this gives \( i^z = e^{z \cdot 5i\pi/4} \) another entire function. **TRUE.**
(b) \( f(z) = e^{i \log(z)} \). Principal branch is \( f(z) = e^{i \log(z)} \).
\( \log(z) \) is analytic at \( z = i \) hence \( f(z) \) is, too. **TRUE.**
(c) \( f(z) = \sinh(z) = \frac{1}{2}(e^z - e^{-z}) \).
\( f(z+2\pi i) = \sinh(z+2\pi i) = \frac{1}{2}(e^{z+2\pi i} - e^{-(z+2\pi i)}) = \frac{1}{2}(e^z - e^{-z}) = \sinh(z) = f(z) \).
**TRUE.**
(d) Cauchy's Integral Formula for \( f(z) = \pi \) gives \( z = f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(w)}{w-z} \, dw \)
for \( z \) inside \( \Gamma \). If \( z \) is outside \( \Gamma \) the function \( \frac{w}{w-z} \)
has no pole in a domain containing \( \Gamma \) so \( \oint_{\Gamma} \frac{w}{w-z} \, dw = 0 \). **TRUE.**
(e) \( \frac{1+z^{10}}{z^{100}} = \frac{z^{-100} + z^{-90}}{z^{100}} \) has an antiderivative, for example \( -\frac{1}{99} z^{-99} - \frac{1}{89} z^{-89} \).
So the integral over every closed contour vanishes. **TRUE.**
Problem 4. (4 points)
Compute the contour integral
\[ \int_{\Gamma} z \, dz, \]
where \( \Gamma \) is the circle of radius 2, centered at \( z = i \), traversed once in the clockwise direction. Simplify your answer.

Parameterize the circle:
\[ z(t) = i + 2e^{-2\pi it}, \quad 0 \leq t \leq 1. \]

\[
\int_{\Gamma} z \, dz = \int_{0}^{1} \left( i + 2e^{-2\pi it} \right) \left( i + 2e^{-2\pi it} \right) dt
\]

\[
= \int_{0}^{1} \left( -i + 2e^{2\pi it} \right) \left( -4\pi i e^{2\pi it} \right) dt
\]

\[
= -4\pi i \int_{0}^{1} e^{-2\pi it} dt - 8\pi i \int_{0}^{1} dt
\]

\[
= -4\pi i \left[ -\frac{1}{2\pi i} e^{-2\pi it} \right]_{0}^{1} - 8\pi i \left[ t \right]_{0}^{1}
\]

\[
= \frac{4\pi i}{2\pi i} (1 - 1) - 8\pi i (1 - 0)
\]

\[= -8\pi i \]
Problem 5. (5 points)
Compute the contour integral
\[ \int_{i}^{-i} \frac{z}{(z+1)^2} dz, \]
along the straight path from \( i \) to \(-i\). Simplify your answer.

\[ f(z) = \frac{z}{(z+1)^2} = \frac{A}{(z+1)^2} + \frac{B}{z+1} \]

\[ A = \lim_{z \to -1} f(z)(z+1)^2 = \lim_{z \to -1} z = -1 \]

\[ B = \lim_{z \to -1} \frac{d}{dz}f(z)(z+1)^2 = \lim_{z \to -1} 1 = 1 \]

Check: \( \frac{-1}{(z+1)^2} + \frac{1}{z+1} = \frac{-1 + z + 1}{(z+1)^2} = \frac{z}{(z+1)^2} = f(z) \) \( \checkmark \)

antiderivative of \( \frac{-1}{(z+1)^2} = -(z+1)^{-2} \) is \( (z+1)^{-1} = \frac{1}{z+1} \)

antiderivative of \( \frac{1}{z+1} \) is \( \log(z+1) \). The branch cut of \( \log(z+1) \) is where \( z+1 = -r \) \( (r > 0 \text{ real}) \) \( \Rightarrow \) \( z = -1 - r \) \( \Rightarrow \) \( z \leq -1 \) real.

Since the path avoids the branch cut, we can use \( \log(z+1) \) as antiderivative of \( \frac{1}{z+1} \).

\[
\begin{align*}
\int_{i}^{-i} \frac{z}{(z+1)^2} dz &= \int_{i}^{-1} \frac{-1}{(z+1)^2} dz + \int_{-1}^{i} \frac{1}{z+1} dz \\
&= \frac{1}{-i+1} - \frac{1}{i+1} + \log(-i+1) - \log(i+1) \\
&= \frac{(1+i) - (1-i)}{(1+i)(1-i)} + \log|1-i| + i \arg(1-i) - (\log|1+i| + i \arg(1+i)) \\
&= \ldots
\end{align*}
\]
Overflow space.

\[ \ldots = \frac{2 \pi}{2} + \ln r^2 + i \left( -\frac{\pi}{4} \right) - \left( \ln r^2 + i \frac{\pi}{4} \right) \]

\[ = i - \frac{\pi}{4} i - \frac{\pi}{4} i \]

\[ = \left( 1 - \frac{\pi}{2} \right) i \]
Problem 6. (5 points)
Compute the contour integral

\[ \oint_{\Gamma} \frac{z^3}{(z-1)^3} \, dz, \]

where \( \Gamma \) is a simple closed curve winding around \( z = 1 \) once in the clockwise direction. Simplify your answer.

By the residue theorem for rational functions,

\[ \oint_{\Gamma} \frac{z^3}{(z-1)^3} \, dz = -2\pi i \quad \text{res} \left( \frac{z^3}{(z-1)^3}; 1 \right) \]

\[ \text{clockwise} \]

\[ = -2\pi i \quad \lim_{z \to 1} \frac{d}{dz} \left( \frac{z^3}{(z-1)^3} \right) \]

\[ = -2\pi i \quad \lim_{z \to 1} \frac{6z}{(z-1)^3} \]

\[ = -6\pi i \]

Alternatively, use

\[ f''(z) = \frac{z^3}{2\pi i} \oint_{\Gamma} \frac{f(w)}{(w-z)^3} \, dw \]

with \( f(w) = w^3 \) and \( z = 1 \):

\[ f''(1) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{w^3}{(w-1)^3} \, dw \]

\[ f'(w) = 3w^2 \]

\[ f''(w) = 6w \]

\[ f''(1) = 6 \]

but the path goes the wrong way, so

\[ \oint_{\Gamma} \frac{z^3}{(z-1)^3} \, dz = -6\pi i \]