Midterm Exam I

October 1, 20124

No books. No notes. No calculators. No electronic devices of any kind.

Name ____________________ **Student Number** ____________________

Problem 1. (3 points)
Find all solutions in $\mathbb{C}$ of the equation

$$z^2 - (i+1)z + i = 0 .$$

Write your answers in the form $z = x + yi$, with x and y real.

Quadratic formula: $z = \frac{i+1}{2} \pm \frac{1}{2}\sqrt{(i+1)^2 - 4i}$

$= \frac{i+1}{2} \pm \frac{1}{2}\sqrt{-1+2i+1-4i}$

$= \frac{i+1}{2} \pm \frac{1}{2}\sqrt{-2i}$

Any square root of $-2i$ will do: the other solution comes from the "$\pm$" in the formula. $-2i = 2e^{-i\frac{\pi}{2}}$ so a square root is $\sqrt{2}\,e^{-i\pi/4}$

$= \sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = 1-i$. Plug in:

$$z = \frac{i+1}{2} \pm \frac{1}{2}(1-i) = \begin{cases} \frac{i+1}{2} + \frac{1-i}{2} = 1 \\ \frac{i+1}{2} - \frac{1-i}{2} = i \end{cases}$$

So the two solutions are $z_1 = 1$

$z_2 = i$

1	2	3	4	5	6	total/25

Problem 2. (3 points)
Find all solutions in $\mathbb{C}$ of the equation

$$z^5 = -32\,.$$

Write your solutions in the form $z = re^{i\theta}$, with r and θ real, $r \geq 0$.

$$z^5 = -32 = 2^5(-1) = 2^5\, e^{i\pi}$$

So $z = 2\, e^{i\pi/5} \sqrt[5]{1}$

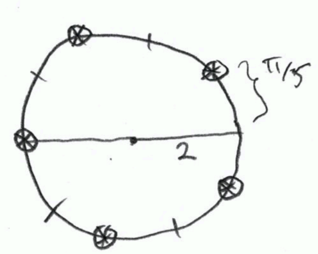
$$= 2\, e^{i\pi/5}\, e^{i\,2\pi k/5} \qquad k=0,1,2,3,4.$$

$$= 2e^{\frac{i\pi}{5}(1+2k)} \qquad k=0,1,2,3,4.$$

The solutions are

$$2e^{i\pi/5},\quad 2e^{i3\pi/5},\quad 2e^{i\pi},\quad 2e^{i7\pi/5},\quad 2e^{i9\pi/5}$$

Sketch:



Another way to write the solutions: $-2,\ e^{\pm i3\pi/5},\ e^{\pm i\pi/5}$.

Problem 3. (5 points)
True or false? (No reasons necessary.)
(a) The function $f(z) = \operatorname{Im}(z) - i\operatorname{Re}(z)$ is analytic in $\mathbb{C}$.
(b) For every $z \in \mathbb{C}$, the real number $\operatorname{Im}(z)$ is a possible value for $\arg(e^z)$.
(c) The complex exponential function maps vertical lines to rays eminating from (but not containing) the origin, and it maps horizontal lines to circles centred at the origin.
defined on the domain $D \subset \mathbb{C}$
(d) If an analytic function takes only pure imaginary values, it is necessarily constant. in D.
(e) The derivative of $f(z) = e^{i\pi z}$ is $f'(z) = \pi e^{i\pi(z+\frac{1}{2})}$.

(a) $f(z) = \frac{z-\bar{z}}{2i} - i\,\frac{z+\bar{z}}{2} = \frac{1}{2}\left(-iz + \cancel{i\bar{z}} - iz - \cancel{i\bar{z}}\right) = -iz$ is analytic TRUE

(b) $e^z = e^{\operatorname{Re}(z) + i\operatorname{Im}(z)} = \underbrace{e^{\operatorname{Re}(z)}}_{|e^z|}\ e^{i\,\underbrace{\operatorname{Im}(z)}_{\arg(e^z)}}$ TRUE

(c) vertical lines: $a+ti \rightsquigarrow e^{a+ti} = e^a(\cos t + i\sin t)$ circle of radius e^a
horizontal lines: $t+bi \rightsquigarrow e^{t+bi} = e^t(\cos b + i\sin b)$ ray at angle b from origin.
So it's just the other way around. FALSE

(d) $f = u+iv$. $u=0$ implies $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ by the Cauchy-Riemann equations.
This implies that v (hence f) is constant if D is a domain. TRUE

(e) $f(z) = e^{i\pi z}$

$f'(z) = i\pi\, e^{i\pi z}$ by the chain rule

$= e^{i\pi/2}\,\pi\, e^{i\pi z} = \pi\, e^{i\pi(z+1/2)}$ TRUE.

Problem 4. (4 points)
Does the following limit exist? If so, compute it.

$$\lim_{n\to\infty}\left(\frac{1}{2}+\frac{i}{2}\right)^n$$

Justify your answer.

$$\left|\frac{1}{2}+\frac{i}{2}\right| = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \sqrt{1/2} < 1$$

so $\lim\limits_{n\to\infty}\left|\frac{1}{2}+\frac{i}{2}\right|^n \to 0$

so $\lim\limits_{n\to 0}\left|\left(\frac{1}{2}+\frac{i}{2}\right)^n - 0\right| \to 0$

so $\lim\limits_{n\to 0}\left(\frac{1}{2}+\frac{i}{2}\right)^n = 0$

Saying that $\left|\frac{1}{2}+\frac{i}{2}\right| < 1$ is the important part.

Problem 5. (5 points)

(a) Find all points $x + iy \in \mathbb{C}$ where the function

$$f(x+iy) = (y^2 - x^2) + \frac{2i}{xy}$$

is complex differentiable.

(b) Find the largest open ~~region~~ set where f is analytic.

Try the Cauchy - Riemann Equations: $u = y^2 - x^2$ $\quad v = \frac{2}{xy} = 2(xy)^{-1}$

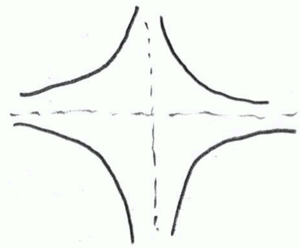
$\frac{\partial u}{\partial x} = -2x \qquad \frac{\partial v}{\partial y} = -2(xy)^{-2}x = \frac{-2}{xy^2} \qquad$ 1 $\quad -2x = \frac{-2}{xy^2} \qquad x^2y^2 = 1$

$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial x} = -2(xy)^{-2}y = \frac{-2}{x^2y} \qquad$ 2 $\quad 2y = \frac{2}{x^2y} \qquad x^2y^2 = 1$

both CR equations give $x^2y^2 = 1$ so $xy = 1$ or $xy = -1$

At all points (x,y) such that $xy = 1$ or $xy = -1$ all partials of u, v are continuous, so f is complex differentiable at all (x,y) where $xy = 1$ or $xy = -1$.

Sketch:



These 4 hyperbolic branches do not contain any open discs (no interior points)

So f is nowhere analytic. The largest open region where f is analytic is empty!

Problem 6. (5 points)

(a) Find a function $v(x, y)$, such that $f = u + iv$ is analytic in $\mathbb{C}$, where u is given as

$$u(x + iy) = x^2 + 2y - y^2 .$$

(b) Express f in terms of z.

[u is harmonic b/c $\frac{\partial^2 u}{\partial x^2} = 2$ $\quad \frac{\partial^2 u}{\partial y^2} = -2$ $\quad$ so $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.]

We need to satisfy the Cauchy Riemann equations: <u>One of the CRE:</u>

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(2-2y) = 2y-2$ $\qquad$ integrate wrt. x:

$v = 2xy - 2x + f(y)$ $\qquad$ so $\qquad \frac{\partial v}{\partial y} = 2x + f'(y)$ $\qquad (*)$

<u>the other CRE:</u>

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x$ $\qquad (*)$

$(*) = (*)$ we get $f'(y) = 0$ so $f(y) = \text{const.}$ We can take $f = 0$.

(a) Then $\quad v(x,y) = 2xy - 2x$.

(b) $f = x^2 + 2y - y^2 + i(2xy - 2x)$

$$= \left(\frac{z+\bar z}{2}\right)^2 + \frac{2(z-\bar z)}{2i} - \left(\frac{(z-\bar z)}{2i}\right)^2 + i\left(2\,\frac{z+\bar z}{2}\,\frac{z-\bar z}{2i} - 2\,\frac{z+\bar z}{2}\right)$$

$$= \frac{z^2}{4} + \cancel{\frac{z\bar z}{2}} + \cancel{\frac{\bar z^2}{4}} - iz + \cancel{i\bar z} + \frac{z^2}{4} - \cancel{\frac{z\bar z}{2}} + \cancel{\frac{\bar z^2}{4}} + \frac{z^2}{2} - \cancel{\frac{\bar z^2}{2}} - iz - \cancel{i\bar z}$$

$$= z^2 - 2iz$$

$$= z(z - 2i)$$

Overflow space.

Another way to do this: $u = \mathrm{Re}(f) = \frac{1}{2}(f+\bar{f})$

So

$$\frac{1}{2}(f+\bar{f}) = x^2 + 2y - y^2 = \left(\frac{z+\bar{z}}{2}\right)^2 + 2\,\frac{z-\bar{z}}{2i} - \left(\frac{z-\bar{z}}{2i}\right)^2$$

$$f+\bar{f} = \frac{z^2}{2} + z\bar{z} + \frac{z\bar{z}}{2} - 2iz + 2i\bar{z} + \frac{\bar{z}^2}{2} - z\bar{z} + \frac{\bar{z}^2}{2}$$

$$f+\bar{f} = z^2 - 2iz + \bar{z}^2 + 2i\bar{z}$$

$$f+\bar{f} = z^2 - 2iz + \overline{z^2 - 2iz}$$

$$\underbrace{f-(z^2-2iz)}_{\text{analytic, by assumption.}} = -\overline{f-(z^2-2iz)}$$

If an analytic function g satisfies $g = -\bar{g}$ or $g+\bar{g} = 0$

or $\mathrm{Re}(g) = 0$, so g takes only pure imaginary values, it is constant.

So $f-(z^2-2iz) = ic$ $\quad c \in \mathbb{R}$ const.

$$\boxed{f = z^2 - 2iz + ic} \quad c \in \mathbb{R}$$