1. (a) \[ z = i \log(-1 + i) = i \left[ \log \sqrt{2} + i \left( \frac{3\pi}{4} + 2n\pi \right) \right], \] so \[ \cos z = \frac{e^{iz} + e^{-iz}}{2} = -\frac{3}{4} + \frac{i}{4}. \]

(b) \[ z = \frac{\sqrt{3}+i}{\sqrt{2}(1+i)} = \frac{(\sqrt{3}+i)(\sqrt{2}-i)}{2\sqrt{2}}. \] Since \( |z| = 1 \) and \( z \) lies in the fourth quadrant, \( \text{Log}(z) = -i \arctan \left( \frac{\sqrt{3}-1}{\sqrt{3}+1} \right) = -i \arctan(2 - \sqrt{3}) \), where \( \arctan \) denotes the inverse tangent function with range in \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \).

(c) \[ \cosh z = \frac{(e^z + e^{-z})}{2}, \] so \( \cosh z = \frac{1}{2} \) implies that \( e^z + e^{-z} = 1 \) or \( e^z = \frac{1+i\sqrt{3}}{2} \). Therefore the solutions are of the form \( z = \log \left( \frac{1+i\sqrt{3}}{2} \right) = \pm \left( \frac{\pi}{3} + 2n\pi \right) \) where \( n \) is any integer.

2. (a) Since \( f \) has continuous first partial derivatives at all points, it is differentiable at all points where Cauchy-Riemann equations hold. Since \( u_x = 1, u_y = 2, v_x = 4(2x - y) \) and \( v_y = -2(2x - y) \), we find that the CR-equations hold if and only if \( 2x - y = -\frac{1}{2} \). Thus \( f \) is differentiable only at the points lying on this line.

(b) Since the line does not contain any open set, \( f \) is analytic nowhere.

(c) Suppose that \( g = u + iw \) is an entire function. By CR equations, \( w_x = -u_y = -2 \) and \( w_y = u_x = 1 \). Therefore, \( w = y - 2x + C \) where \( C \) is any constant. Hence \( g = (1-2i)z+C \) for any arbitrary constant \( C \).

3. The domain of \( f \) indicates that a branch could be defined as follows:

\[ f(z) = \exp \left[ -\frac{1}{2} \mathcal{L}_{-\frac{\pi}{2}}(z-1) \right] \]

where \( \mathcal{L}_{-\frac{\pi}{2}} \) denotes the branch of the complex logarithm with the cut along the nonpositive imaginary axis. In other words, \( \mathcal{L}_{-\frac{\pi}{2}}(z) = \ln |z| + i \arg(z) \), with \( \arg(z) \in (-\frac{\pi}{2}, \frac{3\pi}{2}) \).

Parametrize \( \Gamma \) as \( z(t) = e^{it}, 0 \leq t \leq \pi \). Therefore \( \mathcal{L}_{-\frac{\pi}{2}}(z(t)) = it \), hence

\[ \int_{\Gamma} f(z) \, dz = \int_{0}^{\pi} e^{-\frac{it}{2}} i e^{it} \, dt = i \int_{0}^{\pi} e^{\frac{it}{2}} \, dt = 2(i - 1). \]

4. Use the residue theorem to evaluate all the integrals in this problem.

(a) \( 2\pi i \)

(b) \( -\pi i \)

(c) \( 200\pi i e^{-i} \)

(d) \( -\frac{\pi^2i}{4} \).

5. For any \( K > R \), let \( C_K \) denote the circle centred at \( z_0 = 0 \) with radius \( K \). We make use the inequality for derivatives of analytic functions: for any \( r \geq 1 \),

\[ |f^{(n+r)}(0)| \leq (n+r)! \frac{M_K}{K^{n+r}}, \]

where \( M_K = \sup_{z \in C_K} |f(z)| \). By the hypothesis of this problem, \( M_K \leq CK^n \). Therefore for every \( K > R \), we obtain the estimate

\[ |f^{(n+r)}(0)| \leq (n+r)! \frac{CK^n}{K^{n+r}} = \frac{(n+1)!C}{K^r} \to 0 \text{ as } K \to \infty. \]
Thus $f^{(n+r)}(0) = 0$ for all $r \geq 1$. Now it follows from the Taylor expansion of $f$ that

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}(z - z_0)^j,$$

in other words, $f$ is a polynomial of degree at most $n$.

6. By partial fraction expansion, we find that

$$(1) \quad \frac{1}{(3z - 1)(z + 2)} = \frac{3}{7(3z - 1)} - \frac{1}{7(z + 2)}. $$

(a) For large $|z|$, both of the following inequalities $|1/3z| < 1$ and $|2/z| < 1$. We therefore arrange the expressions above so that the geometric series expansion can be used:

$$\frac{3}{7(3z - 1)} = \frac{1}{7z(1 - \frac{1}{3z})} = \frac{1}{7z} \sum_{k=0}^{\infty} \left(\frac{1}{3z}\right)^k$$

$$\frac{1}{7(z + 2)} = \frac{1}{7z(1 + \frac{2}{z})} = \frac{1}{7z} \sum_{k=0}^{\infty} \left(\frac{2}{z}\right)^k.$$

Therefore for large $|z|$, $f(z) = \frac{1}{7z} \sum_{k=0}^{\infty} \left(3^{-k} - 2^k\right) z^{-k}$.

(b) Here the annular region must be of the form $\{z : r < |z| < R\}$ where $\frac{1}{3} < r < 1 < R < 2$. Thus now $|1/3z| < 1$ and $|z|/2 < 1$, so the second term in (1) has to be arranged differently for the geometric series formula to be applied.

$$\frac{1}{7(z + 2)} = \frac{1}{14(1 + \frac{z}{2})} = \frac{1}{14} \sum_{k=0}^{\infty} \left(-\frac{z}{2}\right)^k.$$

In this region the Laurent series takes the form

$$f(z) = \frac{1}{7z} \sum_{k=0}^{\infty} \left(\frac{1}{3z}\right)^k - \frac{1}{14} \sum_{k=0}^{\infty} \left(-\frac{z}{2}\right)^k.$$

(c) The function $f$ has two simple poles, at $z = \frac{1}{3}$ and $z = -2$ respectively, with $\text{Res}_f(\frac{1}{3}) = \frac{1}{7}$ and $\text{Res}_f(-2) = -\frac{1}{7}$.

(d) $\int_{C} f(z) \, dz = 2\pi i \text{Res}_f(\frac{1}{3}) - 2\pi i \text{Res}_f(-2) = \frac{4\pi i}{7}$. 
7. Expanding $e^{1/z}$ and $1/(1 - z)$ in their Taylor expansions we find that

$$e^{\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{1}{k!z^k} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

$$\frac{1}{1 - z} = \sum_{k=0}^{\infty} z^k = 1 + z + z^2 + z^3 + \cdots,$$

so

$$\text{Res}(e^{\frac{1}{z}} \frac{1}{1 - z}) = \text{coefficient of } \frac{1}{z} \text{ in the product of the two Laurent series}$$

$$= \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$= e - 1.$$