1. (a) Suppose $f$ is a continuous function on $[1, \infty)$. For every real number $t \geq 1$, compute the Riemann-Stieltjes integral

$$F(t) = \int_{1}^{t} f(x) d[x],$$

where $[x]$ is the greatest integer in $x$.

(b) Given a sequence $\{x_n : n \geq 1\}$ of distinct points in $(a, b)$ and a sequence $\{c_n : n \geq 1\}$ of positive numbers with $\sum_{n=1}^{\infty} c_n < \infty$, define an increasing function $\alpha$ on $[a, b]$ by setting

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$$

where $I(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases}$

Show that

$$\int_{a}^{b} f \, d\alpha = \sum_{n=1}^{\infty} c_n f(x_n)$$

for every continuous function $f$ on $[a, b]$.

2. (a) If $f$ and $\alpha$ share a common-sided discontinuity in $[a, b]$, show that $f$ is not in $R_{\alpha}[a, b]$.

(b) Identify the class of functions that are Riemann-Stieltjes integrable on $[a, b]$ with respect to $\alpha$ for every nondecreasing $\alpha$. In other words, describe the set

$$\bigcap \{R_{\alpha}[a, b] : \alpha \text{ nondecreasing} \}.$$

(c) Recall that $S[a, b]$ is the collection of all step functions on $[a, b]$. If $S[a, b] \subseteq R_{\alpha}[a, b]$, show that $\alpha$ is continuous.

3. We have seen $\chi_\mathbb{Q}$ (the indicator function of the rationals) is not Riemann integrable on $[0, 1]$. The problem was that it was too discontinuous - in fact, every point in $[0, 1]$ was a point of discontinuity. Here is another example of a function with uncountably many points of discontinuity, but this time Riemann integrable.

Show that the set of discontinuities of $\chi_\Delta$ (the indicator function of the Cantor middle-third set) is precisely $\Delta$, which is an uncountable set, but that $\chi_\Delta$ is nevertheless Riemann integrable on $[0, 1]$. 