Quiz 2 - Math 105, Section 204

Name: **SID:**

Date: Jan 25, 2012 **Time: 20 mins**

This quiz has two problems. Look overleaf for the second.

---

1. Use the method of Lagrange multipliers to maximize the function $f(x, y) = \frac{3}{2}x^2 + \frac{3}{2}y^2$ subject to the constraint $x^3 + y^3 = 2$. Do not forget to justify why the solution you have obtained gives the max (and not the min).

   (6 + 2 = 8 points)

   Solution: We want to solve the equations

   $$f_x(x, y) = \lambda g_x(x, y),$$

   $$f_y(x, y) = \lambda g_y(x, y),$$

   $$g(x, y) = 0,$$

   that is:

   $$3x = \lambda 3x^2,$$

   $$3y = \lambda 3y^2,$$

   $$y^3 + x^3 = 2.$$

   (3 points)

   Rewriting the first 2 equations gives

   $$3x(1 - \lambda x) = 0, 3y(1 - \lambda y) = 0.$$

   Hence the solutions occur when

   $$x = 0, x = 1/\lambda$$

   and

   $$y = 0, y = 1/\lambda.$$

   If $\lambda \neq 0$, then we get $x = y = 1$. If $\lambda = 0$, then we get $x = y = 0$, but this point isn't on the constraint. If $x = 0$, then $y = \sqrt[3]{2}$, and similarly if $y = 0$ then $x = \sqrt[3]{2}$. Hence the 3 points identified by the Lagrange multiplier method are:

   $$(1, 1), (0, \sqrt[3]{2}), (\sqrt[3]{2}, 0).$$

   The function evaluated at these 3 points is

   $$f(1, 1) = 3,$$

   $$f(0, \sqrt[3]{2}) = f(\sqrt[3]{2}, 0) = \frac{3}{\sqrt[3]{2}}.$$

   (3 points)

The largest of these 3 points is at $(1, 1)$ so if a maximum exists, it occurs at $(1, 1)$. However, there is no maximum, since

$$C^3 + (\sqrt[3]{2 - C^3})^3 = 2$$

for any constant C, hence

$$f(C, \sqrt[3]{2 - C^3}) = \frac{3}{2}\left(C^2 + \sqrt[3]{2 - C^3}^2\right) \geq \frac{3}{2}C^2$$

can become arbitrarily large.

(2 points)

2. Find all the critical points of the function

$$f(x, y) = x^3 - 3xy + y^3$$

and classify them as local maximum, local minimum, saddle point or none of these.

(7 points)

The partial derivatives are

$$f_x = 3x^2 - 3y, f_y = 3y^2 - 3x,$$
$$f_{xx} = 6x, f_{xy} = f_{yx} = -3, f_{yy} = 6y.$$

(1 point)

Solving for $f_x = 0$, we have

$$3x^2 = 3y,$$

and substituting in $f_y = 0$ gives

$$3(x^4 - x) = 3x(x^3 - 1) = 0.$$

(2 points)

Hence $x = 0, 1$. If $x = 0$, then $y = 0$, and if $x = 1$, then $y = 1$. Therefore there are two critical points:

$$(0, 0), (1, 1).$$

(2 points)

For $(0, 0)$, the second derivative test gives

$$D = f_{xx}f_{yy} - f_{xy}^2 = 0 - 9 = -9,$$

so it is a saddle point.

(1 point)

For $(1, 1)$, the second derivative test gives

$$D = f_{xx}f_{yy} - f_{xy}^2 = 36 - 9 = 27,$$

so since $f_{xx} = 6 > 0$, it is a minimum.

(1 point)