Homework 2 - Math 105, Section 204
Due at the beginning of lecture on February 17, 2012

Name: **SID:**

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1. A curve in the (x, y) plane is defined by the equation

$$y = \int_0^{g(x)} e^{x^2 - t^2}\, dt \quad \text{where} \quad g(0) = 1.$$

If the curve has slope -3 at the point $x = 0$, find the value of $g'(0)$.

(10 points)

First, we rewrite the equation as follows:

$$y = e^{x^2} \int_0^{g(x)} e^{-t^2}\, dt.$$

This is valid by either checking that both versions give the same value for fixed x, or by noting that x is independent with respect to t, so it can be treated as a constant. In either case, we are now in a position to take the derivative. If we write

$$F(x) = \int_0^{x} e^{-t^2}\, dt,$$

we see that

$$y' = 2xe^{x^2} F(g(x)) + e^{x^2} F'(g(x))g'(x).$$

Hence as given in the problem, when $x = 0$ we have

$$-3 = F'(g(0))g'(0) = F'(1)g'(0).$$

Therefore we need to determine $F'(1)$ to determine $g'(0)$. To evaluate $F'(1)$, we apply the Fundamental Theorem of Calculus, which says

$$F'(x) = \frac{d}{dx} \int_0^{x} e^{-t^2}\, dt = e^{-x^2},$$

so we get

$$-3 = e^{-1} g'(0),$$

that is

$$g'(0) = -3e.$$

2. Evaluate the indefinite integral

$$\int \frac{x^5 + x^3}{x^4 + 1}\, dx.$$

(Hint: Write $x^5 + x^3 = x(x^4 + 1) + x^3 - x$.)

(10 points)

Solution: Following the hint, we can rewrite the integral as follows:

$$\begin{aligned}\int \frac{x^5 + x^3}{x^4 + 1}\, dx &= \int \frac{x(x^4 + 1)}{x^4 + 1}\, dx + \int \frac{x^3}{x^4 + 1}\, dx - \int \frac{x}{x^4 + 1}\, dx \\ &= \int x\, dx + \int \frac{x^3}{x^4 + 1}\, dx - \int \frac{x}{x^4 + 1}\, dx.\end{aligned}$$

The first integral can be evaluated directly. We get

$$\int x\, dx = \frac{1}{2}x^2 + C.$$

To evaluate the second integral, we use the substitution $u = x^4 + 1$. In that case, we get $du = 4x^3\, dx$, so the integral becomes

$$\frac{1}{4}\int \frac{1}{u}\, du = \frac{1}{4}\log u + C = \frac{1}{4}\log|x^4 + 1| + C.$$

Note: we can replace the absolute values by brackets since $x^4 + 1$ is non-negative. To evaluate the last integral, we use the substitution $u = x^2$. We then get $du = 2x\, dx$, so the integral becomes

$$\frac{1}{2}\int \frac{1}{u^2 + 1}\, du = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan(x^2) + C.$$

Combining our three answers gives

$$\frac{1}{2}x^2 + \frac{1}{4}\log|x^4 + 1| - \frac{1}{2}\arctan(x^2) + C.$$

3. Use integration by parts to find an antiderivative of $f(x) = \sec^4 x$.

(10 points)

To integrate by parts, we pick $u = \sec^2 x$ and $dv = \sec^2 x$, so that $du = 2\sec^2 x \tan x$ and $v = \tan x$. Hence

$$\int f(x)\,dx = \sec^2 x \tan x - 2\int \tan^2 x \sec^2 x\,dx.$$

Using the identity $\tan^2 x = \sec^2 x - 1$, we get

$$\begin{aligned}\int f(x)\,dx &= \sec^2 x \tan x - 2\int (\sec^2 x - 1)\sec^2 x\,dx \\ &= \sec^2 x \tan x - 2\int \sec^4 x\,dx + 2\int \sec^2 x\,dx \\ &= \sec^2 x \tan x + 2\tan x - 2\int f(x)\,dx.\end{aligned}$$

Therefore

$$3\int f(x)\,dx = \sec^2 x \tan x + 2\tan x,$$

that is

$$\int f(x)\,dx = \frac{1}{3}\sec^2 x \tan x + \frac{2}{3}\tan x + C.$$

We also have the alternate solutions

$$\int f(x)\,dx = \sec^2 x \tan x - \frac{2}{3}\tan^3 x + C,$$

$$\int f(x)\,dx = \frac{1}{3}\tan^3 x + \tan x + C.$$