Definite integral as an area: Example 2

Find the value of the integral

$$\int_0^4 (4 - x)\, dx.$$

A. 8

B. 4

C. 2

D. 1

Definite integral as an area: Example 3

Find the value of the integral

$$\int_0^{2\pi} \sin x \, dx.$$

A. 1

B. 0

C. 2π

D. -2π

Approximating areas under curves

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$$\mathcal{R}_n(f) = \sum_{k=1}^{n} f(x_k)\frac{b-a}{n},$$

where x_k is some point in the subinterval $I_k = [a + (k-1)\frac{b-a}{n}, a + k\frac{b-a}{n}]$.

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- The value of $\mathcal{R}_n(f)$ depends on the choice of x_k within I_k.
- If $x_k =$ left (resply right) endpoint of I_k for all k, then $\mathcal{R}_n(f)$ is called the left (resply right) Riemann sum.

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Relation between Riemann sums and the definite integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \mathcal{R}_n(f) = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\frac{b-a}{n}.$$

An example

Which one of the definite integrals below equals the limit

$$\lim_{n\to\infty} \frac{\frac{k}{n^2}}{1+\frac{k^2}{n^2}}?$$

A.

$$\int_1^2 \frac{1}{1+x^2}\,dx$$

B.

$$\int_0^1 \frac{x}{1+x^2}\,dx$$

C.

$$\int_0^1 \frac{1}{1+x^2}\,dx$$

D.

$$\int_1^2 \frac{x^2}{1+x^2}\,dx$$