Consider the function \( f : [0, 2] \to \mathbb{R} \) defined by

\[
f(x) = \begin{cases} 
\sqrt{1-(x-1)^2}, & 0 \leq x \leq 1, \\
x, & 1 < x \leq 2.
\end{cases}
\]

(a) Sketch the graph of the function \( f \) (2 pts).

(b) Find an antiderivative of \( \sqrt{1-(x-1)^2} \) (2 pts). **Hint:** use the formula

\[
\int \sqrt{1-x^2}dx = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2} + C.
\]

**Solution:** We want to do a substitution to get this in the form of the hint. Let \( u = x - 1 \), so \( du = dx \), and applying the substitution rule,

\[
\int \sqrt{1-(x-1)^2}dx = \int \sqrt{1-u^2}du = \frac{u\sqrt{1-u^2}}{2} + \frac{\sin^{-1} u}{2} + C
\]

Substituting in \( u = x - 1 \), we have

\[
\int \sqrt{1-(x-1)^2}dx = \frac{(x-1)\sqrt{1-(x-1)^2}}{2} + \frac{\sin^{-1}(x-1)}{2} + C
\]
(c) Calculate the left Riemann sum for a regular partition and \( n = 4 \). Does the result underestimate or overestimate \( \int_0^2 f(x)dx \) (2+1 = 3 pts)?

**Solution:** The first thing to do is write down the regular partition for the interval \([0, 1]\) and \( n = 4 \).

The points creating the partition are \( x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2 \) and the intervals of the partition are \([0, \frac{1}{2}), [\frac{1}{2}, 1), [1, \frac{3}{2}), [\frac{3}{2}, 2]\).

A Riemann sum for the partition is \( \sum_{i=1}^{4} f(y_i)(x_i - x_{i-1}) \), where \( y_i \) is a point in the closed interval \([x_{i-1}, x_i]\). The left Riemann sum is the Riemann sum where the function is evaluated at the left endpoints, so \( y_i = x_{i-1} \) for each \( i \).

Evaluating at these left endpoints, we have \( f(0) = \sqrt{1 - (0 - 1)^2} = 0, f\left(\frac{1}{2}\right) = \sqrt{1 - \left(\frac{1}{2} - 1\right)^2} = \frac{\sqrt{3}}{2}, f(1) = \sqrt{1 - (1 - 1)^2} = 1, f\left(\frac{3}{2}\right) = \frac{3}{2} \).

Our left Riemann sum is now
\[
\sum_{i=1}^{4} f(x_{i-1})(x_i - x_{i-1}) = 0 \cdot \left(\frac{1}{2} - 0\right) + \frac{\sqrt{3}}{2} \cdot \left(1 - \frac{1}{2}\right) + 1 \cdot \left(\frac{3}{2} - 1\right) + \frac{3}{2} \cdot \left(2 - \frac{3}{2}\right) = \frac{\sqrt{3} + 5}{4}
\]

Looking at the graph we sketched, the function is strictly increasing, so we know that the left Riemann sum will underestimate the integral \( \int_0^2 f(x)dx \).
(d) Calculate the right Riemann sum for a regular partition and \( n = 4 \). Does the result underestimate or overestimate \( \int_0^2 f(x) dx \) (2+1 = 3 pts)?

**Solution:** The partition in this question is the same as in part (c). The right Riemann sum is the Riemann sum where the function is evaluated at the right endpoints, so \( y_i = x_i \) for all \( i \).

Evaluating at the right endpoints, we have \( f\left(\frac{1}{2}\right) = \sqrt{1 - (\frac{1}{2} - 1)^2} = \frac{\sqrt{3}}{2} \), \( f(1) = \sqrt{1 - (1 - 1)^2} = 1 \), \( f\left(\frac{3}{2}\right) = \frac{3}{2} \), \( f(2) = 2 \).

Our right Riemann sum is now

\[
\sum_{i=1}^{4} f(x_i)(x_i - x_{i-1}) = \frac{\sqrt{3}}{2} \cdot \left(\frac{1}{2} - 0\right) + 1 \cdot \left(1 - \frac{1}{2}\right) + \frac{3}{2} \cdot \left(\frac{3}{2} - 1\right) + 2 \cdot \left(2 - \frac{3}{2}\right) = \frac{\sqrt{3} + 9}{4}
\]

The function is strictly increasing, so we know that the right Riemann sum will overestimate the integral \( \int_0^2 f(x) dx \).
(e) Evaluate \( \int_{0}^{2} f(x) \, dx \) using a geometric argument. **Hint:** the area of a circle with radius \( r \) is \( \pi r^2 \) (2 pts).

**Solution:** From 0 to 1, the function traces a quarter circle of radius 1, so the area under this portion of the function is a quarter of the area of a circle, \( \frac{\pi \cdot 1^2}{4} = \frac{\pi}{4} \).

From 1 to 2, the function traces the line \( y = x \). We can split up the area underneath into two parts to make it easy to find the area, namely square with corners \((1, 1), (1, 0), (2, 0), (2, 1)\), which has area 1, and the triangle on top of it with corners \((1, 1), (2, 2), (2, 1)\), which is half of a \( 1 \times 1 \) square, so it has area \( \frac{1}{2} \). The area under this section is then \( \frac{3}{2} \).

(Alternatively, the area under the function from 1 to 2 forms a trapezoid with parallel sides of lengths \( a = 1 \) and \( b = 2 \), and height \( h = 1 \), so using the area formula \( A = \frac{a+b}{2} \cdot h \) gives the same result.)

Our total area is then \( \frac{\pi + 6}{4} \).
(f) Evaluate \( \int_0^2 f(x) \, dx \) using the fundamental theorem of calculus. **Hint:** split the integral into two integrals (3 pts).

**Solution:** The idea here is to split up the integral into intervals where we know an antiderivative. From part (b), we know an antiderivative on the interval \([0, 1]\), and on the interval \([1, 2]\), the function is just the line \( f(x) = x \), so we know the antiderivative there too.

Working out the first piece,

\[
\int_0^1 f(x) \, dx = \int_0^1 \sqrt{1 - (x-1)^2} \, dx = \left[ \frac{(x-1)\sqrt{1-(x-1)^2}}{2} + \frac{\sin^{-1}(x-1)}{2} \right]_0^1
\]

\[
= \left[ \frac{(1-1)\sqrt{1-(1-1)^2}}{2} + \frac{\sin^{-1}(1-1)}{2} \right] - \left[ \frac{(0-1)\sqrt{1-(0-1)^2}}{2} + \frac{\sin^{-1}(0-1)}{2} \right] = \frac{\pi}{4}
\]

And for the other part, we have

\[
\int_1^2 f(x) \, dx = \int_1^2 x \, dx = \left[ \frac{x^2}{2} \right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2}
\]

Adding these together, we get

\[
\int_0^2 f(x) \, dx = \int_0^1 f(x) \, dx + \int_1^2 f(x) \, dx = \frac{\pi}{4} + \frac{6}{4}.
\]