Math 105 Final

1a. Determine whether the following improper integral is convergent or divergent. If it is convergent, compute the integral.

\[ \int_0^5 \frac{1}{\sqrt{25-x^2}} \, dx. \]

1b. Let \( F(x) = \int_{\sin(x)}^{\cos(x)} e^{-t^2} \, dt. \) Compute \( F'(x) \)

1c. Numerical integration Use the Midpoint Rule with \( n = 3 \) to approximate the following integral:

\[ \int_1^{2.5} (x-1)^2 \, dx \]

1d. Determine whether or not the following limit exists:

\( \lim_{(x,y) \to (5,5)} \frac{x^2 + y^2 - 2yx}{x-y} \)

1e. Consider the quadric surface defined by the equation

\[ z = 4x^2 - 9y^2. \]

Draw level curves corresponding to the values \( z = 0, \pm 1, 2 \). Identify the quadric surface (i.e. is it a paraboloid, hyperboloid, or ellipsoid?)

2. Consider the following Demand and Supply curves:

\[ D(q) = 6 - q \quad S(q) = q^2. \]

Find the equilibrium point \((p_e, q_e)\), and compute the Consumers’ and Producers’ surplus.
3. A company estimates that the income produced at time \( t \) by its factory will equal \( 1000 - 50t \). Find the present value over the next ten years, assuming a 5% interest rate.

4. Evaluate the following indefinite integral:

\[ \int \frac{x^2}{\sqrt{16 - x^2}} \, dx. \]

5. Find the area of the region in the first quadrant bounded by the curve \( y = \sqrt{x} \) and the curve \( y = x^3 \).

6. Let \( X \) be a continuous random variable with the following probability density function:

\[ f(x) = \frac{1}{21} x^2 \, dx, \quad 1 \leq x < 4. \]

Find the corresponding cumulative distribution function, \( F(x) \). Use \( F(x) \) to compute the following probabilities: \( P(2 \leq X) \), \( P(X \leq 3) \).

7. A recently deceased person was found in a room, where the room’s temperature was \( 17\,^\circ C \). According to Newton’s law of cooling, the temperature of the body \( y(t) \) at \( t \) hours after death satisfies the differential equation:

\[ y' = k(17 - y), \]

for some constant \( k \).

Assume that the body’s temperature at the time of death is \( 37\,^\circ C \). Further assume that the it is \( 27\,^\circ C \) after 4 hours. Determine the constant \( k \), and solve the differential equation to find \( y(t) \).

8. Let \( f(x, y) = \sqrt{x^3 + 4xy + y^2x + y^4 - 4} \).

a. Find \( f(1, 2) \), \( f_x(1, 2) \), and \( f_y(1, 2) \).

b. Approximate the change in \( f \) as \( x \) changes from 1 to 1.2 and \( y \) changes from 2 to 1.9.

c. Now assume that \( y = g(x) \) is a function of \( x \) defined implicitly by the equation

\[ f(x, y) = 5. \]

Find the equation of the tangent line to the graph \( y = g(x) \) at \( (x, y) = (1, 2) \).

9. Using the method of Lagrange Multipliers, find the maximum and minimum of \( f(x, y) = 3x^2 - 2y^2 + 2y \), with the constraint \( x^2 + y^2 = 1 \).
Solutions:

1a. The integral does converge, and is equal to \( \frac{\pi}{2} \).

1b. \((-\sin(x))e^{-\cos^2(x)} - (\cos(x))e^{-\sin^2(x)}\).

1c. \(\frac{35}{32}\)

1d. The limit does exist, and is equal to 0

1e. The limit does exist, and is equal to 0

1f. The trace for \(z = 0\) is a pair of lines, \(2x = \pm 3y\).

   The trace for \(z = 1\) is the hyperbola \(1 = \frac{x^2}{(2)^2} - \frac{y^2}{(\frac{3}{2})^2}\).

   The trace for \(z = -1\) is the hyperbola \(1 = \frac{y^2}{(\frac{3}{2})^2} - \frac{x^2}{(2)^2}\).

   The trace for \(z = 2\) is the hyperbola \(1 = \frac{x^2}{(\frac{\sqrt{2}}{2})^2} - \frac{y^2}{(\frac{3}{\sqrt{2}})^2}\). The quadric surface is a hyperbolic paraboloid.

2. \((p_e, q_e) = (2, 4)\) Producers surplus is \(\frac{16}{3}\). Consumers’ surplus is 2.

3. \(\int_{0}^{10} (1000 - 50t)e^{-0.05t} \, dt = 6065.3\)

4. \(8 \arcsin\left(\frac{x}{4}\right) - \frac{1}{2}x\sqrt{16 - x^2}\).

5. \(\frac{5}{12}\).

6. \(F(x) = \frac{1}{63}(x^3 - 1)\).

   \(P(2 \leq X) = 1 - F(2) = \frac{48}{63}\)

   \(P(X \leq 3) = F(3) = \frac{80}{63}\).

7. \(k = \frac{1}{4}\ln(2)\).

   \(y(t) = 17 + 20e^{-\frac{1}{4}\ln(2)t}\).

8a. \(f(1, 2) = 5\)

   \(f_x(1, 2) = \frac{2}{3}\)

   \(f_y(1, 2) = \frac{20}{5} = 4\)

8b. \(df = -\frac{1}{16}\).

8c. \(y - 2 = -\frac{3}{8}(x - 1)\)

9. Solutions are \((0, 1), (0, -1), (\frac{\sqrt{24}}{5}, \frac{1}{5})\), and \((-\frac{\sqrt{24}}{5}, \frac{1}{5})\).

   \(f(0, 1) = 0\)

   \(f(0, -1) = -4\)

   \(f(\pm \frac{\sqrt{24}}{5}, \frac{1}{5}) = \frac{16}{5}\).

So \((0, -1)\) is the minimum and \((\pm \frac{\sqrt{24}}{5}, \frac{1}{5})\) are both maximum.