Math 121 Assignment 9
Due Friday April 9

■ Practice problems:
• Try out as many problems from Sections 9.5–9.6 as you can, with special attention to the ones marked as challenging problems. As a test of your understanding of the material, work out the problems given in the chapter review. You may skip the ones that require computer aid.

■ Problems to turn in:
1. Find the centre, radius and interval of convergence of each of the following power series.
   \[(a) \sum_{n=0}^{\infty} \frac{1 + 5^n}{n!} x^n \quad (b) \sum_{n=1}^{\infty} \frac{(4x - 1)^n}{n^n}.\]
2. Expand
   (a) \(1/x^2\) in powers of \(x + 2\).
   (b) \(x^3/(1 - 2x^2)\) in powers of \(x\).
   (c) \(e^{2x+3}\) in powers of \(x + 1\).
   (d) \(\sin x - \cos x\) about \(\frac{\pi}{4}\).
   For each expansion above, determine the interval on which the representation is valid.
3. Find the sums of the following numerical series.
   \[(a) \sum_{n=0}^{\infty} \frac{(n+1)^2}{\pi^n} \quad (b) \sum_{n=1}^{\infty} \frac{(-1)^n n(n+1)}{2^n} \quad (c) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 2^n},\]
   \[(d) x^3 - \frac{x^9}{3! \times 4} + \frac{x^{15}}{5! \times 16} - \frac{x^{21}}{7! \times 64} + \frac{x^{27}}{9! \times 256} - \cdots\]
   \[(e) 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \frac{x^8}{9!} + \cdots\]
   \[(f) 1 + \frac{1}{2 \times 2!} + \frac{1}{4 \times 3!} + \frac{1}{8 \times 4!} + \cdots\]
4. This problem outlines a strategy for verifying whether a function \(f\) is real-analytic. Recall the \(n\)th order Taylor polynomial of \(f\) centred at \(c\):
   \[P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x - c)^k,\]
   and set \(E_n = f(x) - P_n(x)\).
(a) Use mathematical induction to show that

\[ E_n(x) = \frac{1}{n!} \int_c^x (x - t)^n f^{(n+1)}(t) \, dt, \]

provided \( f^{(n+1)} \) exists on an interval containing \( c \) and \( x \). The formula above is known as Taylor’s formula with integral remainder.

(b) Use Taylor’s formula with integral remainder to prove that \( \ln(1+x) \) is real analytic at \( x = 0 \); more precisely, that the Maclaurin series of \( \ln(1+x) \) converges to \( \ln(1+x) \) for \(-1 < x \leq 1\).