16. Let a circular disk with radius $a$ have centre at point $(a, 0)$. Then the disk is rotated about the $y$-axis which is one of its tangent lines. The volume is:

$$V = 2 \times 2\pi \int_0^{2a} x\sqrt{a^2 - (x - a)^2} \, dx$$

Let $u = x - a$

$$du = dx$$

$$= 4\pi \int_{-a}^{a} (u + a)\sqrt{a^2 - u^2} \, du$$

$$= 4\pi \int_{-a}^{a} u\sqrt{a^2 - u^2} \, du + 4\pi a \int_{-a}^{a} \sqrt{a^2 - u^2} \, du$$

$$= 0 + 4\pi a \left(\frac{1}{2} \pi a^2\right) = 2\pi^2 a^3 \text{ cu. units.}$$

(Note that the first integral is zero because the integrand is odd and the interval is symmetric about zero; the second integral is the area of a semicircle.)

![Diagram showing a circle centered at $(a, 0)$, rotated about the $y$-axis, resulting in a solid of revolution with volume $2\pi^2 a^3$ cu. units.](Fig. 1-16)
29. The region is symmetric about $x = y$ so has the same volume of revolution about the two coordinate axes. The volume of revolution about the $y$-axis is

$$V = 2\pi \int_0^8 x(4 - x^{2/3})^{3/2} \, dx$$

Let $x = 8 \sin^3 u$

$$dx = 24 \sin^2 u \cos u \, du$$

$$= 3072\pi \int_0^{\pi/2} \sin^5 u \cos^4 u \, du$$

$$= 3072\pi \int_0^{\pi/2} (1 - \cos^2 u)^2 \cos^4 u \sin u \, du$$

Let $v = \cos u$

$$dv = -\sin u \, du$$

$$= 3072\pi \int_0^1 (1 - v^2)^2 v^4 \, dv$$

$$= 3072\pi \int_0^1 (v^4 - 2v^6 + v^8) \, dv$$

$$= 3072\pi \left( \frac{1}{5} - \frac{2}{7} + \frac{1}{9} \right) = \frac{8192\pi}{105} \text{ cu. units.}$$
9. The volume between height 0 and height \( z \) is \( z^3 \). Thus

\[
z^3 = \int_0^z A(t) \, dt,
\]

where \( A(t) \) is the cross-sectional area at height \( t \). Differentiating the above equation with respect to \( z \), we get \( 3z^2 = A(z) \). The cross-sectional area at height \( z \) is \( 3z^2 \) sq. units.
11. \( V = 2 \int_0^r (2\sqrt{r^2 - y^2})^2 \, dy \)
   
   \[ = 8 \int_0^r (r^2 - y^2) \, dy = 8 \left( r^2 y - \frac{y^3}{3} \right) \bigg|_0^r = \frac{16r^3}{3} \text{ cu. units.} \]
12. The area of an equilateral triangle of base $2y$ is $\frac{1}{2}(2y)(\sqrt{3}y) = \sqrt{3}y^2$. Hence, the solid has volume

$$V = 2 \int_{0}^{r} \sqrt{3}(r^2 - x^2) \, dx$$

$$= 2\sqrt{3}\left(r^2x - \frac{1}{3}x^3\right)\left|_{0}^{r}\right.$$

$$= \frac{4}{\sqrt{3}}r^3 \text{ cu. units.}$$

Fig. 2-12
13. \( y = x^2, \ 0 \leq x \leq 2, \ y' = 2x. \)

\[
\text{length} = \int_0^2 \sqrt{1 + 4x^2} \, dx \quad \text{Let} \ 2x = \tan \theta \\
2 \, dx = \sec^2 \theta \, d\theta \\
= \frac{1}{2} \int_{x=0}^{x=2} \sec^3 \theta \\
= \frac{1}{4} \left( \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right) \bigg|_{x=0}^{x=2} \\
= \frac{1}{4} \left( 2x\sqrt{1 + 4x^2} + \ln(2x + \sqrt{1 + 4x^2}) \right) \bigg|_0^2 \\
= \frac{1}{4} \left( 4\sqrt{17} + \ln(4 + \sqrt{17}) \right) \\
= \sqrt{17} + \frac{1}{4} \ln(4 + \sqrt{17}) \text{ units.}
\]
14. \[ y = \ln \frac{e^x - 1}{e^x + 1}, \quad 2 \leq x \leq 4 \]

\[ y' = \frac{e^x + 1}{e^x - 1} \cdot \frac{(e^x + 1)e^x - (e^x - 1)e^x}{(e^x + 1)^2} \]

\[ = \frac{2e^x}{e^{2x} - 1}. \]

The length of the curve is

\[ L = \int_2^4 \sqrt{1 + \frac{4e^{2x}}{(e^{2x} - 1)^2}} \, dx \]

\[ = \int_2^4 \frac{e^{2x} + 1}{e^{2x} - 1} \, dx \]

\[ = \int_2^4 \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx = \ln |e^x - e^{-x}| \bigg|_2^4 \]

\[ = \ln \left( e^4 - \frac{1}{e^4} \right) - \ln \left( e^2 - \frac{1}{e^2} \right) \]

\[ = \ln \left( \frac{e^8 - 1}{e^4} \cdot \frac{e^2}{e^4 - 1} \right) = \ln \frac{e^4 + 1}{e^2} \text{ units.} \]
36. \[ S = 2\pi \int_{0}^{1} |x| \sqrt{1 + \frac{1}{x^2}} \, dx \]

\[ = 2\pi \int_{0}^{1} \sqrt{x^2 + 1} \, dx \quad \text{Let } x = \tan \theta \]

\[ dx = \sec^2 \theta \, d\theta \]

\[ = 2\pi \int_{0}^{\pi/4} \sec^3 \theta \, d\theta \]

\[ = \pi \left[ \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]_{0}^{\pi/4} \]

\[ = \pi \left[ \sqrt{2} + \ln(\sqrt{2} + 1) \right] \text{ sq. units.} \]
5. The mass of the plate is

\[
m = 2 \int_0^4 ky\sqrt{4 - y} \, dy \quad \text{Let } u = 4 - y \quad du = -dy
\]

\[
= 2k \int_0^4 (4 - u)u^{1/2} \, du
\]

\[
= 2k \left( \frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right) \bigg|_0^4 = \frac{256k}{15}.
\]

By symmetry, \( M_{x=0} = 0 \), so \( \bar{x} = 0 \).

\[
M_{y=0} = 2 \int_0^4 ky^2 \sqrt{4 - y} \, dy \quad \text{Let } u = 4 - y \quad du = -dy
\]

\[
= 2k \int_0^4 (16u^{1/2} - 8u^{3/2} + u^{5/2}) \, du
\]

\[
= 2k \left( \frac{32}{3}u^{3/2} - \frac{16}{5}u^{5/2} + \frac{2}{7}u^{7/2} \right) \bigg|_0^4 = \frac{4096k}{105}.
\]

Thus \( \bar{y} = \frac{4096k}{105} \cdot \frac{15}{256k} = \frac{16}{7} \). The centre of mass of the plate is \((0, \frac{16}{7})\).

Fig. 4-5
11. Choose axes through the centre of the ball as shown in the following figure. The mass of the ball is

\[ m = \int_{-R}^{R} (y + 2R)\pi(R^2 - y^2) \, dy \]

\[ = 4\pi R \left( R^2 y - \frac{y^3}{3} \right) \bigg|_{0}^{R} = \frac{8}{3}\pi R^4 \text{ kg}. \]

By symmetry, the centre of mass lies along the \(y\)-axis; we need only calculate \(\bar{y}\).

\[ M_{y=0} = \int_{-R}^{R} y(y + 2R)\pi(R^2 - y^2) \, dy \]

\[ = 2\pi \int_{0}^{R} y^2(R^2 - y^2) \, dy \]

\[ = 2\pi \left( \frac{R^2 y^3}{3} - \frac{y^5}{5} \right) \bigg|_{0}^{R} = \frac{4}{15}\pi R^5. \]

Thus \(\bar{y} = \frac{4\pi R^5}{15} \cdot \frac{3}{8\pi R^4} = \frac{R}{10}\). The centre of mass is on the line through the centre of the ball perpendicular to the plane mentioned in the problem, at a distance \(R/10\) from the centre of the ball on the side opposite to the plane.
12. A slice at height $z$ has volume $dV = \pi y^2 \, dz$ and density $kz \, \text{g/cm}^3$. Thus, the mass of the cone is

$$m = \int_0^b k z \pi y^2 \, dz$$

$$= \pi k a^2 \int_0^b z \left(1 - \frac{z}{b}\right)^2 \, dz$$

$$= \pi k a^2 \left[\frac{z^2}{2} - \frac{2z^3}{3b} + \frac{z^4}{4b^2}\right]_0^b$$

$$= \frac{1}{12} \pi k a^2 b^2 \, \text{g.}$$

The moment about $z = 0$ is

$$M_{z=0} = \pi k a^2 \int_0^b z^2 \left(1 - \frac{z}{b}\right)^2 \, dz = \frac{1}{30} \pi k a^2 b^3 \, \text{g-cm.}$$

Thus, $\bar{z} = \frac{2b}{5}$. Hence, the centre of mass is on the axis of the cone at height $2b/5 \, \text{cm}$ above the base.