11. \[ \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = 2 \int_0^{1/2} \frac{dx}{\sqrt{\frac{1}{4} - (x - \frac{1}{2})^2}} \]

\[= 2 \lim_{c \to 0^+} \int_c^{1/2} \frac{dx}{\sqrt{\frac{1}{4} - (x - \frac{1}{2})^2}} \]

\[= 2 \lim_{c \to 0^+} \sin^{-1}(2x - 1) \bigg|_{c}^{1/2} = \pi. \]

The integral converges.
14. \[ \int_0^{\pi/2} \sec x \, dx = \lim_{C \to (\pi/2)^-} \ln |\sec x + \tan x| \bigg|_0^C \]
\[ = \lim_{C \to (\pi/2)^-} \ln |\sec C + \tan C| = \infty. \]

This integral diverges to infinity.
38. Since $0 \leq 1 - \cos \sqrt{x} = 2 \sin^2 \left( \frac{\sqrt{x}}{2} \right) \leq 2 \left( \frac{\sqrt{x}}{2} \right)^2 = \frac{x}{2}$, for $x \geq 0$, therefore

$$\int_0^{\pi^2} \frac{dx}{1 - \cos \sqrt{x}} \geq 2 \int_0^{\pi^2} \frac{dx}{x},$$

which diverges to infinity.
Since $\sin x \geq \frac{2x}{\pi}$ on $[0, \pi/2]$, we have

$$
\int_0^\infty \frac{|\sin x|}{x^2} \, dx \geq \int_0^{\pi/2} \frac{\sin x}{x^2} \, dx \\
\geq \frac{2}{\pi} \int_0^{\pi/2} \frac{dx}{x} = \infty.
$$

The given integral diverges to infinity.

Fig. 5-37
40. Since \( \ln x \) grows more slowly than any positive power of \( x \), therefore we have \( \ln x \leq kx^{1/4} \) for some constant \( k \) and every \( x \geq 2 \). Thus, \( \frac{1}{\sqrt[4]{x \ln x}} \geq \frac{1}{kx^{3/4}} \) for \( x \geq 2 \) and \( \int_2^\infty \frac{dx}{\sqrt[4]{x \ln x}} \) diverges to infinity by comparison with \( \frac{1}{k} \int_2^\infty \frac{dx}{x^{3/4}} \).
4. \[ \int_{1}^{\infty} \frac{dx}{x^2 + \sqrt{x} + 1} \] Let \( x = \frac{1}{t^2} \)
\[ dx = -\frac{2}{t^3} dt \]
\[ = \int_{1}^{0} \frac{1}{\left( \frac{1}{t^2} \right)^2 + \sqrt{\frac{1}{t^2}} + 1} \left( -\frac{2}{t^3} \right) dt \]
\[ = 2 \int_{0}^{1} \frac{t \, dt}{t^4 + t^3 + 1}. \]
\[ \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \]

Let \( \sin x = u^2 \)

\[ 2u \, du = \cos x \, dx = \sqrt{1 - u^4} \, dx \]

\[ = 2 \int_0^1 \frac{u \, du}{u \sqrt{1 - u^4}} \]

\[ = 2 \int_0^1 \frac{du}{\sqrt{(1-u)(1+u)(1+y^2)}} \]

Let \( 1 - u = v^2 \)

\[ -du = 2v \, dv \]

\[ = 4 \int_0^1 \frac{v \, dv}{v \sqrt{(1+1-v^2)(1+(1-v^2)^2)}} \]

\[ = 4 \int_0^1 \frac{dv}{\sqrt{(2-v^2)(2-2v^2+v^4)}}. \]
3. One possibility: let \( x = \sin \theta \) and get

\[
I = \int_{-1}^{1} \frac{e^x}{\sqrt{1-x^2}} \, dx = \int_{-\pi/2}^{\pi/2} e^{\sin \theta} \, d\theta.
\]

Another possibility:

\[
I = \int_{-1}^{0} \frac{e^x}{\sqrt{1-x^2}} \, dx + \int_{0}^{1} \frac{e^x}{\sqrt{1-x^2}} \, dx = I_1 + I_2.
\]

In \( I_1 \) put \( 1 + x = u^2 \); in \( I_2 \) put \( 1 - x = u^2 \):

\[
I_1 = \int_{0}^{1} \frac{2e^{u^2-1}u}{u\sqrt{2-u^2}} \, du = 2 \int_{0}^{1} \frac{e^{u^2-1}}{\sqrt{2-u^2}} \, du
\]

\[
I_2 = \int_{0}^{1} \frac{2e^{1-u^2}u}{u\sqrt{2-u^2}} \, du = 2 \int_{0}^{1} \frac{e^{1-u^2}}{\sqrt{2-u^2}} \, du
\]

so \( I = 2 \int_{0}^{1} \frac{e^{u^2-1} + e^{1-u^2}}{\sqrt{2-u^2}} \, du \).
8. Let

\[ I = \int_{1}^{\infty} e^{-x^2} \, dx \]

Let \( x = \frac{1}{t} \)

\[ dx = -\frac{dt}{t^2} \]

\[ = \int_{1}^{0} e^{-(1/t)^2} \left( -\frac{1}{t^2} \right) \, dt = \int_{0}^{1} \frac{e^{-1/t^2}}{t^2} \, dt. \]

Observe that

\[ \lim_{t \to 0^+} \frac{e^{-1/t^2}}{t^2} = \lim_{t \to 0^+} \frac{t^{-2}}{e^{1/t^2}} \left[ \frac{\infty}{\infty} \right] \]

\[ = \lim_{t \to 0^+} \frac{-2t^{-3}}{e^{1/t^2}(-2t^{-3})} \]

\[ = \lim_{t \to 0^+} \frac{1}{e^{1/t^2}} = 0. \]

Hence,

\[ S_2 = \frac{1}{3} \left( \frac{1}{2} \right) \left[ 0 + 4(4e^{-4}) + e^{-1} \right] \]

\[ \approx 0.1101549 \]

\[ S_4 = \frac{1}{3} \left( \frac{1}{4} \right) \left[ 0 + 4(16e^{-16}) + 2(4e^{-4}) + 4 \left( \frac{16}{9}e^{-16/9} \right) + e^{-1} \right] \]

\[ \approx 0.1430237 \]

\[ S_8 = \frac{1}{3} \left( \frac{1}{8} \right) \left[ 0 + 4 \left( 64e^{-64} + \frac{64}{9}e^{-64/9} + \frac{64}{25}e^{-64/25} + \frac{64}{49}e^{-64/49} \right) + 2 \left( 16e^{-16} + 4e^{-4} + \frac{16}{9}e^{-16/9} \right) + e^{-1} \right] \]

\[ \approx 0.1393877. \]

Hence, \( I \approx 0.14 \), accurate to 2 decimal places. These approximations do not converge very quickly, because the fourth derivative of \( e^{-1/t^2} \) has very large values for some values of \( t \) near 0. In fact, higher and higher derivatives behave more and more badly near 0, so higher order methods cannot be expected to work well either.
14. Let \( y = f(x) \). We are given that \( m_1 \) is the midpoint of \([x_0, x_1]\) where \( x_1 - x_0 = h \). By tangent line approximate in the subinterval \([x_0, x_1]\),

\[
f(x) \approx f(m_1) + f'(m_1)(x - m_1).
\]

The error in this approximation is

\[
E(x) = f(x) - f(m_1) - f'(m_1)(x - m_1).
\]

If \( f''(t) \) exists for all \( t \) in \([x_0, x_1]\) and \( |f''(t)| \leq K \) for some constant \( K \), then by Theorem 11 of Section 4.9,

\[
|E(x)| \leq \frac{K}{2} (x - m_1)^2.
\]

Hence,

\[
|f(x) - f(m_1) - f'(m_1)(x - m_1)| \leq \frac{K}{2} (x - m_1)^2.
\]

We integrate both sides of this inequality. Noting that \( x_1 - m_1 = m_1 - x_0 = \frac{1}{2}h \), we obtain for the left side

\[
\left| \int_{x_0}^{x_1} f(x) \, dx - \int_{x_0}^{x_1} f(m_1) \, dx - \int_{x_0}^{x_1} f'(m_1)(x - m_1) \, dx \right|
\]

\[
= \left| \int_{x_0}^{x_1} f(x) \, dx - f(m_1)h - f'(m_1) \frac{(x - m_1)^2}{2} \right|_{x_0}^{x_1}
\]

\[
= \left| \int_{x_0}^{x_1} f(x) \, dx - f(m_1)h \right|.
\]

Integrating the right-hand side, we get

\[
\int_{x_0}^{x_1} \frac{K}{2} (x - m_1)^2 \, dx = \frac{K}{2} \frac{(x - m_1)^3}{3} \Big|_{x_0}^{x_1}
\]

\[
= \frac{K}{6} \left( \frac{h^3}{8} + \frac{h^3}{8} \right) = \frac{K}{24} h^3.
\]

Hence,

\[
\left| \int_{x_0}^{x_1} f(x) \, dx - f(m_1)h \right|
\]

\[
= \left| \int_{x_0}^{x_1} [f(x) - f(m_1) - f'(m_1)(x - m_1)] \, dx \right|
\]

\[
\leq \frac{K}{24} h^3.
\]
13. \[ I = \int_0^1 x^2 \, dx = \frac{1}{3}. \quad M_1 = \left( \frac{1}{2} \right)^2 (1) = \frac{1}{4}. \] The actual error is \[ I - M_1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}. \]

Since the second derivative of \( x^2 \) is 2, the error estimate is

\[ |I - M_1| \leq \frac{2}{24} (1 - 0)^2 (1^2) = \frac{1}{12}. \]

Thus the constant in the error estimate for the Midpoint Rule cannot be improved; no smaller constant will work for \( f(x) = x^2 \).
46. \( \Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt \).

a) Since \( \lim_{t \to \infty} t^{x-1}e^{-t/2} = 0 \), there exists \( T > 0 \) such that \( t^{x-1}e^{-t/2} \leq 1 \) if \( t \geq T \). Thus
\[
0 \leq \int_T^\infty t^{x-1}e^{-t} \, dt \leq \int_T^\infty e^{-t/2} \, dt = 2e^{-T/2}
\]
and \( \int_T^\infty t^{x-1}e^{-t} \, dt \) converges by the comparison theorem.

If \( x > 0 \), then
\[
0 \leq \int_0^T t^{x-1}e^{-t} \, dt < \int_0^T t^{x-1} \, dt
\]
converges by Theorem 2(b). Thus the integral defining \( \Gamma(x) \) converges.

b) \( \Gamma(x + 1) = \int_0^\infty t^x e^{-t} \, dt \)
\[
= \lim_{c \to 0+} \int_c^R t^x e^{-t} \, dt
\]
\[
= \lim_{c \to 0+} \left( t^x \left[ e^{-t} \right]_c^R + x \int_c^R t^{x-1} e^{-t} \, dt \right)
\]
\[
= 0 + x \int_0^\infty t^{x-1} e^{-t} \, dt = x\Gamma(x).
\]

c) \( \Gamma(1) = \int_0^\infty e^{-t} \, dt = 1 = 0! \).

By (b), \( \Gamma(2) = 1\Gamma(1) = 1 \times 1 = 1 = 1! \).

In general, if \( \Gamma(k+1) = k! \) for some positive integer \( k \), then
\( \Gamma(k+2) = (k+1)\Gamma(k+1) = (k+1)k! = (k+1)! \).

Hence \( \Gamma(n+1) = n! \) for all integers \( n \geq 0 \), by induction.

d) \( \Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2}e^{-t} \, dt \)

Let \( t = x^2 \)
\[
dt = 2x \, dx
\]
\[
= \int_0^\infty \frac{1}{x} e^{-x^2} 2x \, dx = 2 \int_0^\infty e^{-x^2} \, dx = \sqrt{\pi}
\]
\( \Gamma\left(\frac{3}{2}\right) = \Gamma\left(1\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \sqrt{\pi} \).