Math 121 Assignment 2
Due Friday January 22

■ Practice problems (do NOT turn in):
• Try out as many problems from Sections 5.5, 5.6, 5.7 as you can, with special attention to the ones marked as challenging problems. As a test of your understanding of the material, work out the problems given in the chapter review. You may skip the ones that require computer aid.

■ Problems to turn in:
1. Find the function \( f \) that satisfies the equation
\[
2f(x) + 1 = 3 \int_x^1 f(t) \, dt.
\]
2. Find the area of
   (a) the plane region bounded between the two curves \( y = \frac{4}{x^2} \) and \( y = 5 - x^2 \).
   (b) Find the area of the closed loop of the curve \( y^2 = x^4(2 + x) \) that lies to the left of the origin.
3. Evaluate the integrals
   (a) \( \int_0^4 \sqrt{9t^2 + t^4} \, dt \).
   (b) \( \int \cos^2 \left( \frac{t}{5} \right) \sin^2 \left( \frac{t}{2} \right) \, dt \).
   (c) \( \int \cos^4 x \, dx \).
   (d) \( \int \frac{dx}{e^x + 1} \).
   (e) \( \int_0^2 \frac{x \, dx}{x^4 + 16} \).
   (f) \( \int_0^{\pi/2} \sqrt{1 - \sin(\theta)} \, d\theta \).
4. Use mathematical induction to show that for every positive integer \( k \),
\[
\sum_{j=1}^n j^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + P_{k-1}(n),
\]
where $P_{k-1}$ is a polynomial of degree at most $k - 1$. Deduce from this that
\[
\int_0^a x^k \, dx = \frac{a^{k+1}}{k+1}.
\]

5. Does the function
\[
F(x) = \int_0^{2x-x^2} \cos \left( \frac{1}{1+t^2} \right) \, dt
\]
have a maximum or minimum value? Justify your answer.

6. Find the maximum value of
\[
\int_a^b (4x - x^2) \, dx
\]
for intervals $[a, b]$, where $a < b$.

7. (a) If $m, n$ are integers, compute the integrals
\[
\int_{-\pi}^\pi \cos mx \cos nx \, dx, \quad \int_{-\pi}^\pi \sin mx \sin nx \, dx, \quad \int_{-\pi}^\pi \sin mx \cos nx \, dx.
\]
(b) Suppose that for some positive integer $k$, \[f(x) = \frac{a_0}{2} + \sum_{n=1}^k (a_n \cos nx + b_n \sin nx)\]
holds for all $x \in [-\pi, \pi]$. Find integral formulas for the coefficients $a_0, a_n, b_n$ with the integrand involving $f$ of course.

(Remark: The coefficients $a_0, a_n, b_n$ are called the Fourier coefficients of $f$. Fourier coefficients arise in a variety of contexts, such as communications and signal processing. If $f$ is a musical note, then the integers $n$ for which $a_n$ or $b_n$ are nonzero are precisely the frequencies comprising the note.)