1. In 1927, Schauder initiated the formal theory of bases in Banach spaces by offering up a basis for $C[0, 1]$ that now bears his name. The purpose of this problem is to understand his construction.

Consider the dyadic rationals in $[0, 1]$, i.e.,
\[
\{r_{jk} = \frac{k}{2^j} : (j, k) \in \mathbb{Z}^2, j \geq 0, 0 \leq k \leq 2^j\}.
\]
Enumerate these rationals according to the lexicographic order in $(j, k)$ avoiding repetitions, so that
\[
t_0 = 0, \quad t_1 = 1, \quad t_2 = \frac{1}{2}, \quad t_3 = \frac{1}{4}, \quad t_4 = \frac{3}{4}, \quad \ldots.
\]
Let $f_0 \equiv 1$, $f_1(t) = t$. For $n \geq 2$, and $t_n = k_n2^{-jn}$ with $\gcd(k_n, 2) = 1$, define $f_n$ to be the continuous, piecewise linear, tent-shaped function that vanishes outside $[t_n - 2^{-jn}, t_n + 2^{-jn}]$, and whose graph within this interval is given by the two lines joining the points $(t_n - 2^{-jn}, 0)$ with $(t_n, 1)$ and $(t_n, 1)$ with $(t_n + 2^{-jn}, 0)$ respectively. (Drawing a few pictures may help.)

(a) Show that the set $\{f_n : n \geq 1\}$ is linearly independent. (Hint: Observe that $f_n(t_n) = 1$ and $f_k(t_n) = 0$ for $k > n$.)

(b) Show that the span $\{f_0, \ldots, f_{2^m}\}$ is the set of all continuous piecewise linear or “polygonal” functions with nodes at the dyadic rationals $\{k2^{-m} : k = 0, 1, \ldots, 2^m\}$.

(c) It remains to check that $\{f_n : n \geq 1\}$ is a Schauder basis for $C[0, 1]$. How does one show that a countably infinite linearly independent set in a Banach space is a basic sequence? The following test for Schauder bases, due to Banach, is extremely useful:

**Theorem 1.** A sequence $\{x_n : n \geq 1\}$ of nonzero vectors is a Schauder basis for the Banach space $X$ if and only if
(i) \(\{x_n : n \geq 1\}\) has dense linear span in \(X\), and
(ii) there is a constant \(K > 0\) such that

\[
\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq K \left\| \sum_{i=1}^{m} a_i x_i \right\|
\]

for all scalars \(\{a_i\}\) and all \(n < m\).

We will soon be able to prove this result, but assuming it for now, show that \(\{f_n\}\) is a Schauder basis for \(C[0,1]\).

(d) In light of part (c), each \(f \in C[0,1]\) can be uniquely written as a uniformly convergent series \(f = \sum_{k=0}^{\infty} a_k f_k\). Describe the approximating polygonal functions, i.e., the partial sums of this expansion, in terms of \(f\).

(e) It is tempting to wonder whether the monomials \(\{t^n : n = 0, 1, 2, \cdots\}\) might form a Schauder basis for \(C[0,1]\). Do they?

3. (a) If \(n \geq 1\), show that there is a measure \(\mu\) on \([0,1]\) such that \(p'(0) = \int p d\mu\) for every polynomial \(p\) of degree at most \(n\).
(b) Does there exist a measure $\mu$ on $[0, 1]$ such that $p'(0) = \int p \, d\mu$ for every polynomial $p$?

4. In class, we proved that the Fourier transform is an isometric isomorphism from $L^2[0, 2\pi]$ onto $\ell^2(\mathbb{Z})$. An ingredient of the proof was the observation that the space of continuous functions on $[0, 2\pi]$ is dense in $L^2[0, 2\pi]$. In this problem, we investigate this issue in greater generality.

(a) Let $X$ be a locally compact Hausdorff space equipped with a Radon measure $\mu$. Recall that $C_c(X)$ is the space of all $\mathbb{F}$-valued continuous functions on $X$ with compact support. Show that $C_c(X)$ is dense in $L^p(X)$ for $1 \leq p < \infty$. [Hint: One method of proof uses the following measure-theoretic result, known as Lusin’s theorem (look up the proof in Folland’s Real Analysis or Rudin’s Real and Complex Analysis, if you do not know it already):

Theorem 2. Let $X$ be as above, $A$ a measurable subset of $X$ with $\mu(A) < \infty$, and suppose $f$ is an $\mathbb{F}$-valued measurable function on $X$ such that $f(x) = 0$ if $x \notin A$. Given any $\epsilon > 0$, there exists $g \in C_c(X)$ such that

$$\mu \left( \{ x : f(x) \neq g(x) \} \right) < \epsilon.$$  

The function $g$ may be chosen to further satisfy

$$\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|.$$  

You may use this result without proof.]

(b) If $X = \mathbb{R}^d$, $d \geq 1$, the result above may be strengthened as follows. Let $C^\infty_c(\mathbb{R}^d)$ denote the space of infinitely differentiable functions of compact support. Show that $C^\infty_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$. Prove this.

(c) The result that we needed for our proof (of the isometry of the Fourier transform) was that

$$C = \{ f \in C[0, 2\pi] : f(0) = f(2\pi) \}$$

is dense in $L^2[0, 2\pi]$. Explain why this follows from the results above.

(d) Do these approximation theorems hold for $p = \infty$?