1. Here is an application of the notion of weak/weak* convergence in probability theory.

(a) Prove Helly’s selection principle. Namely, let \( \{\mu_n\} \) be a sequence of probability measures on \([0, 1]\). Then there exists a probability measure \( \mu \) and a subsequence \( \{\mu_{n_k} : k \geq 1\} \) such that for all \( f \in C[0, 1] \)

\[
\int f \, d\mu_{n_k} \rightarrow \int f \, d\mu \quad \text{as} \quad k \to \infty.
\]

(b) Given any sequence of numbers \( \{a_n : n \in \mathbb{Z}\} \), how can we determine whether these numbers occur as the Fourier coefficients of some probability measure on \([-\pi, \pi]\)? The key idea here is positive definiteness.

A doubly infinite sequence \( \{a_m : m \in \mathbb{Z}\} \) of complex numbers is said to be a positive definite sequence if for each \( n = 1, 2, \ldots \), the \( n \times n \) matrix \( A_n = \{(a_{i-j})\} \), \( 0 \leq i, j \leq n-1 \) constructed from this sequence is positive semidefinite, i.e., for all \( N \geq 1 \) and all \( z \in \mathbb{C}^N \),

\[
\sum_{n,m=1}^{N} a_{n-m}z_n\bar{z}_m \geq 0.
\]

Show that if \( \mu \) is a probability measure, then the sequence

\[
a_n = \int e^{-inx} \, d\mu(x)
\]

is positive definite, in the sense described above.

(c) Prove the converse of the statement in part (b), originally due to Herglotz. More precisely, let \( \{a_n : n \in \mathbb{Z}\} \) be a positive definite sequence and suppose \( a_0 = 1 \). Then show that there exists a
probability measure $\mu$ on $[-\pi, \pi]$ such that

$$a_n = \int_{-\pi}^{\pi} e^{-inx} d\mu(x).$$

2. We have seen that for a convex set $K$ in a Banach space $X$, the norm closure of $K$ equals the weak closure of $K$. Is this statement always true if $K$ is a convex subset of $U = X^*$ (for some Banach space $X$) and “weak” is replaced by “weak*”? 

3. Is $U = X^*$ equipped with the weak* topology metrizable?

4. Show that every normed linear space $X$ is isometric to a subspace of $C(K)$ for some compact Hausdorff space $K$. If $X$ is separable, show that $K$ can be chosen to be a compact metric space.

5. The previous exercise suggests that the spaces $C(K)$ (of continuous functions on a given compact Hausdorff space $K$) deserve special attention, containing as they do isometric copies of every normed linear space. Let us catalog some important properties of the spaces $C(K)$ for different choices of $K$.

   Remember the Cantor middle-third set (we will denote this by $\Delta$)? The goal of this problem is to uncover the “universal” nature of $C(\Delta)$. More precisely, we will prove that $C(\Delta)$ is the “biggest” among the spaces $C(K)$, where $K$ is a compact metric space. We will do this by showing that every compact metric space $K$ is the continuous image of $\Delta$.

   (a) Convince yourself that the statement above is right, i.e., if $\varphi : \Delta \to K$ is a continuous surjection, then there exists a linear isometry from $C(K)$ to $C(\Delta)$.

   (b) Show that $[0, 1]$ is a continuous image of $\Delta$, as is the cube $[0, 1]^N$.

   (c) Recalling that the elements of $\Delta$ are sequences of 0-s and 2-s, i.e., $\Delta = \{0, 2\}^N$, let us endow $\Delta$ with the following natural metric:

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{3^n},$$

where $\{a_n\}$ and $\{b_n\}$ are the sequences of digits (0-s and 2-s) occurring in the ternary expansion of $x$ and $y$ respectively. Convince yourself (but you need not submit a solution) that $d$ is equivalent to the usual metric on $\Delta$. Moreover, $d$ has the additional property that $d(x, y) = d(x, z)$ implies that $y = z$.

In subsequent discussions, take the metric on $\Delta$ to be the one described above.

Show that every compact metric space is homeomorphic to a closed subspace of $[0, 1]^N$. 

(d) Deduce that every compact metric space $K$ is the continuous
image of $\Delta$. In light of part (a) of this problem, we now know
that $C(K)$ is isometric to a closed subspace of $C(\Delta)$.

6. OK, we have just now seen that $C(\Delta)$ is “universal” for the class
of spaces $C(K)$, for compact metric spaces $K$. In particular, $C[0, 1]$
sits inside $C(\Delta)!$ This line of argument must seem backward, given
how much more publicity $C[0, 1]$ receives than $C(\Delta)$ (think of your
first course in analysis). Let’s ask whether $C[0, 1]$ can be universal
too, i.e., whether $C(\Delta)$ embeds isometrically into $C[0, 1]$.

(a) Given a function $f \in C(\Delta)$, define an extension $\tilde{f}$ of $f$ as a
continuous function on $[0, 1]$ so that the extension map $E(f) = \tilde{f}$
from $C(\Delta)$ into $C[0, 1]$ is a linear isometry.

(b) Deduce that $C(\Delta)$ is isometric to a complemented subspace of
$C[0, 1]$. (A subspace $M$ is complemented in a Banach space $X$
if $M$ is the range of a continuous linear projection $P$ on $X$).

(c) Pull together the facts compiled in problems 4-6 to prove the
Banach-Mazur theorem: every separable normed linear space is
isometric to a subspace of $C[0, 1]$.

7. Prove the spectral radius formula stated in class.

8. Find the spectra of the left shift operator $L : \ell^2 \rightarrow \ell^2$ defined by
$$L(a_0, a_1, a_2, \cdots) = (a_1, a_2, \cdots),$$
and the Volterra operator $V : C[0, 1] \rightarrow C[0, 1]$ defined by
$$Vx(s) = \int_0^s x(r) \, dr.$$