Math 421/510, Spring 2008, Final Exam
(Due date: Monday April 21)

Instructions
• The final exam should be submitted to the instructor’s mailbox by 5 pm on Monday April 21. There will be no extensions for the final.
• Unlike homework assignments, you must work on the final on your own. If you need hints or clarifications, please feel free to talk to the instructor.
• Answers should be clear, legible, and in complete English sentences. Solutions must be self-contained – only results proved in class can be used without proof.

1. Let $X$ be a closed subspace of the Hilbert space $L^2[0, 1]$, and assume that every element of $X$ is essentially bounded (i.e., belongs to $L^\infty[0, 1]$). Prove that $\dim X < \infty$.

2. An important class of bounded linear operators on Hilbert spaces that arises in the study of integral equations is the class of compact operators. Given two Hilbert spaces $H_1$ and $H_2$, an operator $K : H_1 \to H_2$ is said to be compact if the image under $K$ of the unit ball in $H_1$ has compact closure in $H_2$. Let $\mathcal{K}(H_1, H_2)$ denote the class of compact operators.
(a) Show that $\mathcal{K}(H_1, H_2)$ is a closed subset of $\mathcal{B}(H_1, H_2)$.
(b) Given a complex valued function $a(t)$ which is continuous on $[a, b]$, let $A : L^2[a, b] \to L^2[a, b]$ be the bounded linear operator given by

$$(Af)(t) = a(t)f(t).$$

Characterize all functions $a \in C[a, b]$ for which $A$ is compact.

3. Let $\mathfrak{X}$ be a normed space. Recall that the convex hull of any subset of $\mathfrak{X}$ is the smallest convex set containing the given set. Show that the norm closure of the convex hull of any subset $S$ of $\mathfrak{X}$ contains the weak sequential closure of $S$. (An equivalent way of formulating this result is as follows: Let $\{x_n\}$ be a sequence in a normed vector space $\mathfrak{X}$ that converges weakly to $x$. Show that for every $\varepsilon > 0$ and $m \in \mathbb{N}$, there is a convex combination $y = \sum_{n \geq m} \lambda_n x_n$, i.e., a finite sum with non-negative coefficients and $\sum_{n \geq m} \lambda_n = 1$, such that $||x - y|| < \varepsilon$.)