1. Let \( \{\varphi_1, \cdots, \varphi_n\} \) be an orthonormal set in a separable Hilbert space \( \mathcal{H} \), and let \( \{\psi_1, \cdots, \psi_n\} \subseteq \text{span}\{\varphi_1, \cdots, \varphi_n\}^\perp \). For \( h \in \mathcal{H} \), define the finite rank operator \( K \) as follows,
\[
Kh = \sum_{i=1}^n \langle h, \varphi_i \rangle \psi_i.
\]
Prove that \( I + \alpha K \) is invertible for any \( \alpha \) and find its inverse.

2. (a) (The Schur test) Given a doubly infinite matrix \( A = ((\alpha_{nm})) \) and a sequence \( \{a_n\} \subset \mathbb{R}_+ \) such that
\[
\sum_{n=1}^\infty |\alpha_{nm}|a_n \leq ba_m \quad \text{for every } m
\]
\[
\sum_{m=1}^\infty |\alpha_{nm}|a_m \leq ca_n \quad \text{for every } n
\]
for suitable constants \( b \) and \( c \). Show that there is \( T \in \mathcal{B}(\mathcal{H}) \) with \( A \) as its matrix, and that \( ||T||^2 \leq bc \).

(b) (The Hilbert matrix) Show that there exists \( T \in \mathcal{B}(\mathcal{H}) \) whose matrix relative to an orthonormal basis \( \{e_n : n \in \mathbb{N}\} \) is given by \( A = ((\alpha_{nm})) \), with \( \alpha_{nm} = (n + m - 1)^{-1} \). Show that \( T = T^* \) and that \( ||T|| \leq \pi \).

Hint: Use the Schur test with \( a_n = (n - \frac{1}{2})^{-1/2} \).

3. Recall the multiplication operator \( M_\phi \) on \( L^2[a, b] \) that we discussed in class, \( \phi \in L^\infty[a, b] \).
(a) Can \( M_\phi \) be a compact operator for \( \phi \neq 0 \)?
(b) Describe the spaces \( \ker(M_\phi) \) and \( \text{Ran}(M_\phi) \), where \( \phi \in L^\infty[a, b] \) and \( M_\phi \) is the multiplication operator on \( L^2[a, b] \) that we discussed in class.
(c) Suppose that \( \phi \) is a polynomial. Find a necessary and sufficient condition on \( \phi \) for \( \text{Ran}(M_\phi) \) to be closed.
4. In this problem, we continue our study of the unilateral shift operators, namely the right shift \( S_r \) and the left shift \( S_l \) on \( \ell^2(\mathbb{N}) \).

(a) For \( S_r \), find \( S_r^* \).

(b) Show that \( S_r \) has no eigenvalues, but that every \( \lambda \in \mathbb{C} \) with \( |\lambda| < 1 \) is an eigenvalue for \( S_r^* \) with multiplicity 1.

(c) Show that none of the eigenvalues of \( S_r^* \) are orthogonal to each other.

(d) Prove that for \( |\mu| < 1 \), both \( I - \mu S_r \) and \( I - \mu S_l \) are invertible. What happens when \( |\mu| \geq 1 \)?

(e) For \( |\mu| > 1 \), describe \( \ker(I - \mu S_l) \) and \( \text{Ran}(I - \mu S_r)^\perp \), and find their dimensions.

5. In this problem, we explore another application of functional-analytic techniques to solving integral equations. Given \( k \in L^2([a, b] \times [a, b]) \) and \( g \in L^2[a, b] \), consider the integral equation

\[
 f(t) - \int_a^b k(t, s)f(s)ds = g(t).
\]

Here “\( = \)” means equal almost everywhere.

(a) Show that if \( ||k||_{L^2} < 1 \), there exists a unique solution \( f \in L^2[a, b] \) of the above integral equation.

(b) (The earlier version of this problem was poorly worded. Sorry about the confusion.) If in addition

\[
 \sup_t \int_a^b |k(t, s)|^2ds = c < \infty,
\]

find an explicit expression for the solution \( f \) of (1) in terms of the given data \( g \). In particular, identify \( f \) as an integral operator acting on \( g \), and estimate its operator norm in terms of \( ||k||_2 \) and the constant \( c \) in (2).

(c) Find an explicit solution of the integral equation

\[
 f(t) - \lambda \int_0^1 e^{\lambda(t-s)}f(s)ds = g(t) \in L^2[0, 1].
\]

6. Show that the collection of all finite-rank operators on a Hilbert space \( \mathcal{H} \) is a minimal ideal of \( \mathcal{B}(\mathcal{H}) \) (i.e., it is an ideal that does not contain any nontrivial ideals).