Solution for Q.1.

(a) \( I \subset \bigcup_{\alpha \in A} G_{\alpha} \).

Claim : \( X \) is non-empty.
\( \exists \alpha' \) such that \( a \in G_{\alpha'} \) and \( G_{\alpha'} \) is open. Thus, \( a \) is an interior point of \( G_{\alpha'} \) and \( a \in (c, d) \subset G_{\alpha'} \). Thus, if \( a < x < d \), then \( x \in X \). This proves \( X \) is non-empty.

Claim : \( \sup X = b \).

Suppose not, \( \sup(X) = x_o < b \) and \( x_0 \in [a, b] \subset \bigcup_{\alpha \in A} G_{\alpha} \). This implies, \( \exists \alpha'' \in A \) such that \( x_0 \in (c', d') \subset G_{\alpha''} \) and \( d' < b' \). Since \( c' < x_0 \), it is not a upper bound of \( X \). \( \exists x > c' \) such that \( x \in X \), i.e, \([a, x]\) has a finite subcover. Let \([a, x] \subset \bigcup_{\alpha \in B} G_{\alpha} \), where \( B \) is a finite subcover of \( A \). Then

\[ [a, d') \subset \bigcup_{\alpha \in B} G_{\alpha} \cup (c', d'). \]

Choose \( x_0 < y < d' \), then \( y \in X \). This is a contradiction to the fact that \( \sup(X) = x_0 \).

Claim : \( b \in \sup(X) \).

\( b \) is an interior point in the open cover of \([a, b]\). Thus, \( \exists B \in A \) such that \( b \in I_B = (c_1, d_1) \subset G_B \). Since \( b = \sup(X) \), \( \exists x \in (c_1, b) \) such that \( x \in X \). This implies \( I_x \) has a finite subcover. Adding \((c_1, d_1)\) to that cover implies \( b \in X \). This proves \( I \) is compact.

(b) Claim : \( X \) is non-empty.

By (a), \( a_1 \times [a_2, b_2] \) is compact. Thus, every open cover has a finite subcover. Let \( a_1 \times [a_2, b_2] \subset \bigcup_{i=1}^{n} U_{\alpha_i} \). There exists \( r > 0 \) such that \((a_1, a_1 + r) \times [a_2, b_2] \subset \bigcup_{i=1}^{n} U_{\alpha_i} \).

Thus \( X \) is non-empty.

Claim : \( \sup(X) = b_1 \).

Similar to (a), suppose \( \sup(X) = x_0 \). Similar to (a), since \( x_0 \) is an interior point we can arrive at a contradiction as \( x_0 + r \) will also lie in \( X \). The same argument prove that \( b_1 \) lies in \( x \).

(c) We can use induction on \( k \). If \( k - 1 \) is compact, then the finite subcover of \( a_1 \times [a_2, b_2] \times [a_3, b_3] \times \ldots \times [a_k, b_k] \) is used to prove that \( x \) is non-empty.

Solution for Q.2.
(a) Choose $\epsilon = 1$, $\exists x_1, x_2, \ldots, x_n$ such that

$$A \subset \bigcup_{i=1}^{n} N_\epsilon(x_i)$$

Let $M = \sup_{i \leq i \leq N} |x_i|$. If $y \in A$, $\exists i$ such that $y \in N_\epsilon(x_i)$, then

$$|y| \leq |y - x_i| + |x_i| \leq M + 1.$$ 

Hence $A$ is bounded. However the converse isn’t true. Take $M = \{x_n : x_n$ is bounded $\}$, i.e., $M$ is the set of all bounded sequences. If $x = \{x_n\}$ and $y = \{y_n\}$. $d(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n|$. Choose $A = \{e^n : \text{All entries of } e^n \text{ are zero except } n^{th}\}$.

$$d(e^n, e^m) = 1 \text{ for all } n \neq m$$

and $|e^n| = 1$. If we choose $\epsilon = 1/2$, we conclude that $A$ is not totally bounded but bounded.

(b) For all $\epsilon$, $\bigcup_{x \in A} N_\epsilon(x)$ is an open cover of $A$. Since, $A$ is compact, $\exists$ a finite subcover, i.e., $\exists x_1, x_2, \ldots, x - n \in A$ such that $A \subset \bigcup_{i=1}^{n} N_\epsilon(x_i)$.

(c) Take $M = \mathbb{R}$ and $A = \mathbb{Q} \cap [0, 1]$. Clearly $A$ is not compact. Since $[0, 1]$ is compact, it is totally bounded by (b). Thus for every $\epsilon$, $\exists x_1, x_2, \ldots, x - n \in M$ such that

$$A \subset [0, 1] \subset \bigcup_{i=1}^{n} N_\epsilon(x_i).$$

Solution for Q.3. Take

$$\{0, 1, 0, \frac{1}{2}, \frac{2}{2}, 0, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, 0, \frac{1}{4}, \ldots\}$$

Let $r = p/q$. The subsequence $p/q, 2p/2q, 3p/3q, \ldots$ converges to $r$. If $r$ is an irrational number, there exist a sequence of rationals $x_n$ converging to $r$. Let $x_1 = p/q, x_2 = p_2/q_2, x_3 = p_3/q_3, \ldots$. The subsequence $r_1 = p_1/q_1, r_2 = q_1p_2/q_1q_2, r_3 = q_1q_2p_3/q_1q_2q_3$ converges to $r$.

Theorem 3.7 of the book says the set of all subsequential limits is closed.

(a) Solution for Q.4. It follows from the midterm question that if $x, y \in C$, then $\exists z$ such that $x < z < y$ and $z \notin C$. Thus, $C$ is totally disconnected.