1. Let \((X, d)\) be a metric space. Suppose that every subset of \(X\) is compact. Prove that \(X\) must be finite.

\textit{Solution}

Let \(x \in X\). Since \(X \setminus \{x\}\) is compact, it is closed, and thus \(\{x\}\) is an open set. Consider the open cover \(\{\{x\} : x \in X\}\) of \(X\). Since \(X\) is compact, there must exist a finite subcover; call this sub-cover \(\{x_1\}, \{x_2\}, \ldots, \{x_n\}\). But this implies that \(X \subset \bigcup_{i=1}^{n} \{x_i\} = \{x_1, \ldots, x_n\}\). We conclude that \(X\) has cardinality at most \(n\), and in particular \(X\) is finite.

2. Let \(S = \{\xi_1, \xi_2, \xi_3\} \subset \mathbb{R}^3: \xi_1^2 + \xi_2^2 + \xi_3^3 = 1\) be the unit sphere in \(\mathbb{R}^3\); we will think of this set as a subset of \(\mathbb{R}^3\), where \(\mathbb{R}^3\) has the usual Euclidean metric.

Let \((C_\infty, d)\) be the extended complex plane from Homework 5, problem 1, with the metric defined in that problem.

Let \(\pi: S^2 \to C_\infty\) be the stereographic projection, and let \(\pi^{-1}\) be its inverse (you proved in HW 5 that \(\pi\) is a bijection, so in particular \(\pi^{-1}\) exists).

(a) Prove that a set \(G \subset S^2\) is relatively open in \(S^2\) if and only if \(\pi(G)\) is open in \(C_\infty\).

(b) Prove that \((C_\infty, d)\) is compact.

\textit{Solution}

(a) Suppose \(G\) is open. If \(z \in G\) then there is a neighbourhood \(N_r(z) \subset G\). Let \(z' = \pi(z)\) be image of \(z\) in \(C_\infty\), and let \(G' = \pi(G)\) be the image of \(G\). By definition of the metric \(d\) on \(C_\infty\), the points in the neighbourhood \(N_r(z') \subset C_\infty\) correspond exactly to the image of \(N_r(z)\). Thus \(N_r(z') \subset G'\), so \(G'\) is an open subset of \(C_\infty\).

The converse works the same way, and we see that \(G\) is open if and only if \(G'\) is open.

(b) Let \(\{G_\alpha\}\) be an open cover of \(C_\infty\). Then the pre-images \(\{\pi^{-1}(G_\alpha)\}\) (as defined in part (1)) form an open cover of \(S^2\). Since \(S^2 \subset \mathbb{R}^3\) is closed and bounded, it is compact (by Heine-Borel). Thus there is a finite subcover \(\{\pi^{-1}(G_{\alpha_1}), \ldots, \pi^{-1}(G_{\alpha_n})\}\). But then \(\{G_{\alpha_1}, \ldots, G_{\alpha_n}\}\) is an open cover of \(C_\infty\), so \(C_\infty\) is compact.

3. Consider the metric space \((C[a,b], d)\) from Homework 5 problem 3. Is this metric space compact? Prove that your answer is correct.

\textit{Solution}

No. We will find a sequence in \(C[a,b]\) that does not have a convergent subsequence. For each \(n \in \mathbb{N}\), define \(d_n = a + (b - a)2^{-n}\). Consider the piecewise linear function

\[ f_n(x) = \begin{cases} 1 - 2^n(x - a), & a \leq x \leq d_n, \\ 0, & d_n \leq x \leq b. \end{cases} \]

By construction, \(f_n(x)\) is continuous, \(f_n(d_n) = 0\), and if \(m < n\) then \(f_m(d_n) \geq 1/2\). Thus if \(m < n\), we have \(\sup_{x \in [a,b]} |f_n(x) - f_m(x)| \geq |f_n(d_n) - f_m(d_n)| \geq 1/2\), so \(d(f_n, f_m) \geq 1/2\). This if \(n \neq m\), \(d(f_n, f_m) \geq 1/2\). We conclude that the sequence \(\{f_n\}\) defined in this fashion does not have a convergent subsequence.

4. Let \((X, d)\) be a metric space and let \(a \in X\), \(B \subset X\). Define \(d(a, B) = \inf\{d(a, b) : b \in B\}\).
(a) Consider $\mathbb{R}^k$ with the Euclidean metric. Let $B \subset \mathbb{R}^k$ be nonempty and compact, and let $a \in B^c$. Prove that there exists $b \in B$ such that $d(a, b) = d(a, B)$.

(b) Consider $\mathbb{R}^k$ with the Euclidean metric. Let $B \subset \mathbb{R}^k$ be nonempty and closed, and let $a \in B^c$. Prove that there exists $b \in B$ such that $d(a, b) = d(a, B)$.

(c) Consider $\mathbb{Q}$ with the usual Euclidean metric $d(p, q) = |p - q|$. Give an example of a nonempty closed subset $B \subset \mathbb{Q}$ and a rational number $a \in B^c$ such that there is no $b \in B$ for which $d(a, b) = d(a, B)$.

Solution

(a) Since $d(a, B) = \inf\{d(a, b) : b \in B\}$, there exists $b_n \in B$ such that $d(a, b_n) < d(a, B) + \frac{1}{n}$, for all $n \in \mathbb{N}$. Let $E = \{b_n : n \in \mathbb{N}\}$. By Theorem 2.41, $E$ has a limit point in $\mathbb{E}$, call it $b' \in E$. Let $m \in \mathbb{N}$ be given. There exists $M \geq m$ such that $d(b', b_M) < \frac{1}{m}$ (we can insist on $M \geq m$ because every neighbourhood of $b'$ contains infinitely many points of $E$ by Theorem 2.20). By the triangle inequality,

$$d(a, b') \leq d(a, b_M) + d(b_M, b') < d(a, B) + \frac{1}{M} + \frac{1}{m} \leq d(a, B) + \frac{2}{m}.$$ 

Since $m$ is arbitrary, this means $d(a, b') \leq d(a, B)$. However, by definition of $d(a, B)$ and the fact that $b' \in B$, $d(a, B) \leq d(a, b')$. Therefore, $d(a, B) = d(a, b')$.

(b) Choose any $b_0 \in B$, and let $r = d(a, b_0) + 1$. Let $B_0 = B \cap \overline{N_r(a)}$. Then $d(a, B_0) \geq d(a, B)$ (the infimum can only be smaller when it is taken over a larger set). The set $B_0$ is closed and bounded, hence compact. By part (a), there exists $b' \in B_0 \subset B$ such that $d(a, b') = d(a, B_0)$. It is enough now to show that $d(a, b') \leq d(a, B)$. This follows from the facts that $d(a, b') = d(a, B_0) \leq d(a, x_0)$ for all $x_0 \in B_0$, and that $d(a, b') \leq d(a, b_0) < r < d(a, x)$ for all $x \in B \setminus B_0$.

(c) Let $a = 2$ and $B = \{p \in \mathbb{Q} : p^2 < 2\}$. The set $B$ is a closed subset of $\mathbb{Q}$ ($\sqrt{2}$ is not an element of $\mathbb{Q}$). However, $d(2, B) = 2 - \sqrt{2}$, while $d(2, p) > 2 - \sqrt{2}$ for every $p \in B$. (It is perfectly fine to use $\sqrt{2}$ in this last argument, since every metric maps into the real numbers.)