Math 440/508 Quiz 8 Solution

Name: SID #: 

1. Explain whether there exists an analytic branch of the logarithm on the domain

\[ \Omega = \mathbb{C} \setminus \{ z = x + ix^2 : x \geq 0 \} . \]

If yes, give an explicit formula for the logarithm.

\[ \text{(10 points)} \]

Solution. The domain \( \Omega \) is simply connected, so we know that there is an analytic branch of the logarithm. The issue is to define the function “\( z \mapsto \arg(z) \)” so that it is continuous on this domain.

The circle \(|z| = r\) intersects the curve \( \{ x + ix^2 : x \geq 0 \} \) at a unique point \( z_0(r) = x_0(r) + i(x_0(r))^2 \), where

\[ x_0^2 + x_0^4 = r^2, \quad \text{i.e.} \quad x_0(r) = \frac{1}{\sqrt{2}} \sqrt{\sqrt{1 + 4r^2} - 1}. \]

Let \( \theta_0(r) \) denote the unique value of \( \arctan(x_0(r)) \) that lies in \([0, \frac{\pi}{2})\).

Given \( z = x + iy \in \Omega \) with \(|z| = r\), we define

\[ \arg_\Omega(z) = \theta, \]

where \( \theta \) is the unique value of \( \arctan(y/x) \) lying in \((\theta_0(r), \theta_0(r) + 2\pi)\). We note that this defines a continuous function on \( \Omega \). An analytic branch of the complex logarithm on \( \Omega \) is then given by

\[ \log_\Omega(z) = \log|z| + i\arg_\Omega(z). \]

\[ \square \]