1. Suppose that $f$ is meromorphic on an open set containing $\overline{D}$, the closure of the unit disk. Assume that $f$ does not vanish on $\partial D$, and that

$$\frac{1}{2\pi i} \oint_{\partial D} g(z) \frac{f''(z)}{f(z)} \, dz = 0$$

for all functions $g$ that are holomorphic on $D$. What can you say about the zeros and poles of $f$ in $D$?

(10 points)

Solution. We will prove that $f$ has no zeros or poles in $D$.

Aiming for a contradiction, let us assume that $Z = \{z_1, \ldots, z_M\}$ and $P = \{p_1, \ldots, p_N\}$ are respectively the distinct zeros and poles of $f$ in $D$. Set $a_j$ (resp. $b_k$) to be the order of $z_j$ (resp. $p_k$). Then there exists an analytic function $F$ not vanishing anywhere on $D$ such that

$$f(z) = \prod_{j=1}^{M} (z - z_j)^{a_j} \prod_{k=1}^{N} (z - p_k)^{-b_k} F(z).$$

As we saw in the proof of the argument principle, this means

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{M} \frac{a_j}{z - z_j} - \sum_{k=1}^{N} \frac{b_k}{z - p_k} + \frac{F'(z)}{F(z)}.$$

Multiplying both sides of the equation above by a holomorphic function $g$ and integrating over $\partial D$, we find that

$$0 = \oint_{\partial D} g(z) \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^{M} \oint_{\partial D} g(z) \frac{a_j}{z - z_j} - \sum_{k=1}^{N} \oint_{\partial D} g(z) \frac{b_k}{z - p_k} + \oint_{\partial D} g(z) \frac{F'(z)}{F(z)}$$

$$= 2\pi i \left[ \sum_{j=1}^{M} a_j g(z_j) - \sum_{k=1}^{N} b_k g(p_k) \right],$$

where the last step follows from Cauchy’s theorem and the Cauchy integral formula.

Now fix an index $j$, and choose $g$ to be a polynomial that vanishes at every point in $Z$ and $P$ except $z_j$. Then the above computation shows that $2\pi i a_j g(z_j) = 0$, which is a contradiction since $a_j$ is by definition a positive integer and $g(z_j) \neq 0$ by our choice of $g$. This shows that $Z = \emptyset$. The proof for $P$ is identical. $\square$