1. Express the integral
\[ \int_{0}^{\pi} \frac{d\theta}{a + \sin^2 \theta} \]
in the form
\[ \oint_{|z|=1} \frac{f(z)}{z-z_0} \, dz, \]
and then use the Cauchy integral formula to evaluate it. Here \( a > 1 \) is a fixed constant.

**Solution.** We write
\[ \int_{0}^{\pi} \frac{d\theta}{a + \sin^2 \theta} = \int_{0}^{\pi} \left[ a + \frac{1}{2} (1 - \cos(2\theta)) \right]^{-1} d\theta = \frac{1}{2} \int_{0}^{2\pi} \left[ a + \frac{1}{2} (1 - \cos \varphi) \right]^{-1} d\varphi \]
\[ = \int_{0}^{2\pi} \left[ (2a + 1) - \cos \varphi \right]^{-1} d\varphi, \]
where the second step uses the change of variable \( \varphi = 2\theta \). Set \( z = e^{i\varphi} \). Then, \( d\varphi = \frac{1}{iz} \, dz \), and
\[ \cos \varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} = \frac{z + 1/z}{2}. \]
Substituting these into the integral above, we obtain
\[ \int_{0}^{\pi} \frac{d\theta}{a + \sin^2 \theta} = \frac{1}{i} \oint_{|z|=1} \frac{2dz}{2(2a+1)z - z^2 - 1} = -\frac{2}{i} \oint_{|z|=1} \frac{1}{z-z_1} \, dz, \]
where
\[ z_0 = 2a + 1 - 2\sqrt{a^2 + a}, \text{ and } z_1 = 2a + 1 + 2\sqrt{a^2 + a} \]
are the two roots of the polynomial \( z^2 - 2(2a+1)z + 1 \). Note that \( z_1 \) lies outside the unit disk and \( z_0 \) lies in its interior. Therefore by the Cauchy integral formula with \( f(z) = \frac{1}{z-z_1} \), we obtain
\[ \int_{0}^{\pi} \frac{d\theta}{a + \sin^2 \theta} = -\frac{2}{i} \frac{\pi i}{z_0 - z_1} = \frac{\pi}{\sqrt{a^2 + a}}. \]
\( \Box \)