Chapter 3, Exercise 14

Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.

Solution

Assume $f$ is an entire injective function. Then $f$ is nonconstant, so $g(z) := f(1/z)$ has either a pole or an essential singularity at $z = 0$. We will show first that the singularity at 0 cannot be an essential singularity. If it were an essential singularity, then the Cazorati-Weierstrass theorem would imply that the set $g(B(0, 1) \setminus \{0\})$ is dense in $\mathbb{C}$. However, $g(B(2, 1/2))$ is an open set by the open mapping theorem. Therefore these two sets intersect, which shows that $g(z)$ and hence $f(z)$ is not injective.

Therefore, the singularity at $z = 0$ must be a pole, implying that $f(z)$ is a polynomial. Suppose $f(z)$ is a polynomial of degree $m$. Then $f$ has $m$ roots, counting multiplicity. Evidently, if $f$ has two distinct roots, then $f$ is not injective. Thus $f(z) = c(z - z_0)^m$ for some complex numbers $c$ and $z_0$. However, for $m \geq 2$ such functions are also non-injective: $f(z_0 + 1) = c = f(z_0 + e^{2\pi i/m})$. Thus $m = 1$ and $f(z)$ is a linear polynomial (evidently $c \neq 0$ since $f$ is nonconstant).

Chapter 3, Exercise 22

Show that there is no holomorphic function $f$ in the unit disc $D$ that extends continuously to $\partial D$ such that $f(z) = 1/z$ for each $z \in \partial D$.

Solution

We will abuse notation a bit and let $f$ be the continuous extension of this function to $\overline{D}$. Notice that $f(z)$ is uniformly continuous on $\overline{D}$ since $\overline{D}$ is compact. By uniform continuity we have

$$\lim_{r \to 1} \int_{|z|=r} f(z) \, dz = \int_{|z|=1} f(z) \, dz.$$

We know that each integral $\int_{|z|=r} f(z) \, dz$ is zero because $f$ is holomorphic inside $D$. However, we also know that

$$\int_{|z|=1} f(z) \, dz = 2\pi i$$

because $f(z) = \frac{1}{z}$ on the circle where $|z| = 1$. This is a contradiction, so no such function $f$ can exist.
Chapter 3, Problem 3

Suppose \( f \) is holomorphic in a region containing the annulus \( \{ z : r_1 \leq |z - z_0| \leq r_2 \} \) where \( 0 < r_1 < r_2 \).

Show that
\[
f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n,
\]
where the series converges absolutely in the interior of the annulus.

Proof

Fix \( z \), and let \( C_1 \) and \( C_2 \) be the inner and outer circles, respectively. Draw two close, parallel line segments \( S_1 \) and \( S_2 \), separated by a distance \( \delta \), connecting \( C_1 \) and \( C_2 \) so that the point \( z \) is not contained in the small region bounded between the line segments \( S_1 \) and \( S_2 \). Without loss of generality suppose that the minor arc between \( S_1 \) and \( S_2 \) runs clockwise from \( S_1 \) to \( S_2 \) along both \( C_1 \) and \( C_2 \).

Consider the contour \( \Gamma \) starting at the intersection point of \( C_2 \) and \( S_1 \), moving counterclockwise along the major arc of \( C_2 \) until running into the intersection of \( S_2 \) and \( C_2 \), then travelling along \( S_2 \) until the intersection of \( S_2 \) and \( C_1 \), then travelling clockwise around \( C_1 \) until reaching the intersection of \( S_1 \) and \( C_1 \), then travelling along \( S_1 \) until reaching the intersection of \( S_1 \) and \( C_2 \). This is a closed contour that encloses \( z \) but not \( z_0 \). By using independence of path together with the Cauchy integral formula, we get
\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z).
\]
By taking the distance \( \delta \) to go to zero, we see by uniform continuity that
\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta.
\]
We will use this identity to write a Laurent series expansion for \( f(z) \) centered at \( z_0 \). First we will consider the \( C_2 \) integral,
\[
\frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta.
\]
In order to expand this integral as a series, we rewrite
\[
\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0 + z_0 - z} = \frac{f(\zeta)}{\zeta - z_0} \left( \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \right) = \frac{f(\zeta)}{\zeta - z_0} \sum_{j=0}^{\infty} \left( \frac{z - z_0}{\zeta - z_0} \right)^j.
\]
When we integrate this over $C_2$, we get

$$\frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z_0} \sum_{j=0}^{\infty} \left( \frac{z - z_0}{\zeta - z_0} \right)^j d\zeta.$$

We would like to interchange the integral and the sum. To do this, we appeal to the dominated convergence theorem: because $|\zeta - z_0|$ is greater than $|z - z_0|$ for $\zeta \in C_2$, It follows that the series inside the integral is absolutely convergent (and bounded above by the value $\left| \frac{f(\zeta)}{\zeta - z_0} \right| \cdot \frac{1}{1 - \left| \frac{z - z_0}{z - z_0} \right|}$), which in turn is bounded above by a constant value not depending on $\zeta$ for $\zeta \in C_2$. Therefore, the dominated convergence theorem applies and we can interchange the order of the integral and the sum.

$$\sum_{j=0}^{\infty} (z - z_0)^j \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z_0} \left( \frac{z - z_0}{\zeta - z_0} \right)^j d\zeta.$$

We thus take

$$a_j = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z_0} \left( \frac{z - z_0}{\zeta - z_0} \right)^j d\zeta$$

giving the positive-degree terms in the Laurent series expansion. The sum is absolutely convergent by an argument similar to the one allowing us to apply the dominated convergence theorem.

A similar strategy for $C_1$ gives the negative-degree terms in the Laurent series expansion:

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0 + z_0 - z} = -\frac{f(\zeta)}{z - z_0} \frac{1}{1 - \frac{z - z_0}{z - z_0}} = -\frac{f(\zeta)}{z - z_0} \sum_{j=0}^{\infty} \left( \frac{z - z_0}{z - z_0} \right)^j$$

Leaving us with the integral

$$-\frac{1}{2\pi i} \int_{C_1} \frac{-f(\zeta)}{z - z_0} \sum_{j=0}^{\infty} \left( \frac{z - z_0}{z - z_0} \right)^j.$$
so, for $k \leq -1$, taking

$$a_k = \frac{1}{2\pi i} \int_{C_1} (\zeta - z_0)^{-(k+1)}$$

gives the desired negative-degree terms in the Laurent series expansion. The sum is absolutely convergent by an estimate similar to the one used to apply the dominated convergence theorem.