Chapter 1, Exercise 5

A set $\Omega$ is said to be pathwise connected if any two points in $\Omega$ can be joined by a (piecewise-smooth) curve contained entirely in $\Omega$. The purpose of this exercise is to prove that an open set $\Omega$ is pathwise connected if and only if $\Omega$ is connected.

Part (a)

Suppose first that $\Omega$ is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where $\Omega_1$ and $\Omega_2$ are disjoint, non-empty open sets. Choose two points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let $\gamma$ denote a curve in $\Omega$ joining $w_1$ to $w_2$. Consider a Parameterization $z : [0, 1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$, and let

\[ t^* = \sup \{ t : z(s) \in \Omega_1 \text{ for all } 0 \leq s < t \}. \]

Arrive at a contradiction by considering the point $z(t^*)$

Solution

As suggested, we consider the point $z(t^*)$. We ask the question: which of $\Omega_1$ and $\Omega_2$ contains this point? Evidently, this point is not in $\Omega_1$: if $z(t^*)$ is in $\Omega_1$, then, because $\Omega_1$ is open, there is an open ball $B$ containing $z(t^*)$. Since $z$ is continuous, it follows that $z^{-1}(B)$ is open as a subset of $[0, 1]$. Thus (assuming $t^* < 1$) $z^{-1}(\Omega_1)$ contains points to the right of $t^*$, which is impossible. If $t^* = 1$, then there is a sequence of points in $\Omega_1$ that converges to $z(1) \in \Omega_2$, contradicting the assumption that $\Omega_2$ is open.

If we assume instead that $z(t^*) \in \Omega_2$, we recognize that $z(t) \in \Omega_2$ if and only if $t > t^*$. Thus $t^*$ is the infimum of all values of $t$ such that $z(t) \in \Omega_2$, and we can use the same argument as in the previous paragraph to conclude $z(t^*) \notin \Omega_2$. Since $z(t^*) \in \Omega_1 \cup \Omega_2$, this is a contradiction.

0.1 Part b

Conversely, suppose that $\Omega$ is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subset \Omega$ denote the set of all points that can be joined to $w$ by a curve contained in $\Omega$. Also, let $\Omega_2 \subset \Omega$ denote the set of all points that cannot be joined to $w$ by a curve in $\Omega$. Prove that both $\Omega_1$, $\Omega_2$ are open, disjoint, and their union is $\Omega$. Finally, since $\Omega_1$ is nonempty (why?) conclude that $\Omega = \Omega_1$ as desired.

0.1.1 Solution

Evidently $\Omega_1 \cup \Omega_2 = \Omega$ and $\Omega_1$ is disjoint from $\Omega_2$. The only thing that remains to be shown is that both $\Omega_1$ and $\Omega_2$ are open.
Let \( w_1 \in \Omega_1 \). Because \( \Omega \) is open, \( \Omega \) contains an open ball \( B \) centered at \( w_1 \). It is obvious that if \( w^* \in B \), then there is a path \( z^* \) connecting \( w_1 \) and \( w^* \). Let \( z_1 \) be a curve joining \( w \) to \( w_1 \). Then consider the curve defined by 

\[
z(t) = \begin{cases} 
z(2t) & \text{if } 0 \leq t < 1/2 \\
z(2t - 1) & \text{if } 1/2 \leq t \leq 1
\end{cases}
\]

Then \( z \) is a continuous, piecewise smooth curve that connects \( w \) to \( w^* \). It follows that \( B \subset \Omega_1 \) and that \( \Omega_1 \) is open.

Now, let \( w_2 \in \Omega_2 \). Because \( \Omega \) is open \( \Omega \) contains an open ball \( B \) centered at \( w_2 \). Let \( w^* \in B \). If there were a curve \( z_2 \) that connected \( w \) to \( w^* \), then we could, as in the previous paragraph, find a curve connecting \( w \) to \( w_2 \) by concatenating the path from \( w \) to \( w^* \) and the path from \( w^* \) to \( w_2 \). Thus \( w_2 \in \Omega_1 \), which is a contradiction.

Thus \( \Omega \) can be written as \( \Omega_1 \cup \Omega_2 \) for disjoint open sets \( \Omega_1 \) and \( \Omega_2 \). Since \( \Omega \) is connected, either \( \Omega_1 = \Omega \) or \( \Omega_2 = \Omega \). But \( w \in \Omega_1 \), so \( \Omega_1 \) is nonempty and therefore \( \Omega_1 = \Omega \).

**Chapter 1, Exercise 7**

The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called Blaschke factors, will reappear in the various applications in later chapters.

**Part a**

Let \( z, w \) be two complex numbers such that \( \bar{z}w \neq 1 \). Prove that

\[
\left| \frac{w - z}{1 - \bar{w}z} \right| < 1 \text{ if } |z| < 1 \text{ and } |w| < 1
\]

and also that

\[
\left| \frac{w - z}{1 - \bar{w}z} \right| = 1 \text{ if } |z| = 1 \text{ or } |w| = 1.
\]

[Hint: Why can we assume that \( z \) is real? It then suffices to prove that

\[
(r - w)(r - \bar{w}) \leq (1 - rw)(1 - r\bar{w})
\]

with equality for appropriate \( r \) and \( |w| \).]
Solution

Write $z = re^{i\theta}$. Then

$$\frac{|w - z|}{1 - \bar{w}z} = \frac{|w - re^{i\theta}|}{1 - \bar{w}re^{i\theta}} \geq \frac{|e^{i\theta} \bar{w}e^{-i\theta} - r|}{1 - \bar{w}e^{-i\theta}r} = \frac{|we^{-i\theta} - r|}{1 - \bar{w}e^{-i\theta}r}.$$

Letting $w^* = we^{-i\theta}$ this becomes

$$\frac{|w^* - r|}{1 - w^*r},$$

so it is enough to consider the case in which $z = r$ is a real number. Note further that replacing $w$ by $\bar{w}$ is equivalent to taking the complex conjugate of the entire fraction. So it is enough to show

$$\left(\frac{w - r}{1 - wr}\right)\left(\frac{\bar{w} - r}{1 - \bar{w}r}\right) \leq 1$$

or equivalently that

$$(w - r)(\bar{w} - r) \leq (1 - wr)(1 - \bar{w}r).$$

Suppose first that $w$ and $r$ both have absolute value less than 1. Let $w = se^{i\theta}$. Pull out $e^{i\theta}$ and $e^{-i\theta}$ from the first and second factor on the left turns the left side into $(s - r)^2$. Doing the same on the right side turns the expression to $(1 - sr)^2$. Since $s, r < 1$, we have that $sr < \min(|s|, |r|) \leq \max(|s|, |r|) < 1$ so $(s - r)^2$ is clearly smaller than $(1 - sr)^2$ and we are done.

If $s$ is instead equal to 1, then $s - r = 1 - r = 1 - sr$, and if $r = 1$, then $s - r = s - 1 = -(1 - s) = -(1 - sr)$, so we have equality in these cases.

Part b

Prove that for a fixed $w$ in the unit disc $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w - z}{1 - wz}$$

satisfies the following conditions:

1. $F$ maps the unit disc to itself (that is $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic
2. $F$ interchanges 0 and $w$, namely $F(0) = w$ and $F(w) = 0$.
3. $|F(z)| = 1$ if $|z| = 1$.
4. $F : \mathbb{D} \to \mathbb{D}$ is bijective. [Hint: Calculate $F \circ F$.]
Solution

(i) and (iii) directly follow from part (a) of the problem except for the holomorphicity, which is clear except when $z\bar{w} = 1$. This can only happen if $|w| = 1$ and $z = \frac{1}{w}$. It is seen that $F$ has a removable singularity at $z = \frac{1}{w}$ with value $w$. (ii) follows by plugging in: the numerator is clearly 0 when $z = w$, and plugging in $z = 0$ makes the numerator equal to $w$ and the denominator equal to 1. All that remains to be seen is that $F$ is bijective on $\mathbb{D}$. Consider $F \circ F(z)$. This is

$$F \circ F(z) = \frac{w - \frac{w - z}{1 - w\bar{z}}}{1 - \frac{w - z}{1 - w\bar{z}}}.$$

We simplify this:

$$= \frac{w - \frac{w - z}{1 - w\bar{z}}}{1 - \frac{w - z}{1 - w\bar{z}}} = \frac{w(1 - \bar{w}z) - (w - z)}{q - \bar{w}z - \bar{w}(w - z)} = \frac{w - |w|^2z - w + z}{1 - \bar{w}z - |w|^2 + \bar{w}z} = \frac{z(1 - |w|^2)}{1 - |w|^2} = z$$

so the function $F$ is an involution and therefore bijective on $\mathbb{D}$.

Chapter 1, Exercise 13

Suppose that $f$ is holomorphic in an open set $\Omega$. Prove that in any one of the following cases:

1. Re($f$) is constant;
2. Im($f$) is constant;
3. $|f|$ is constant; one can conclude that $f$ is constant.

Solution

Suppose that Re($f$) is constant. Then $f(x, y) = a + iv(x, y)$ for $z = x + iy$. Then we consider the PDEs from the Cauchy-Riemann equations:

$$0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$
So \( \frac{\partial v}{\partial y} \) is zero, and thus \( v \) depends only on \( x \) and

\[
0 = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

so \( \frac{\partial v}{\partial x} \) is zero and thus \( v \) depends only on \( y \). Since \( v \) cannot depend on either \( x \) or \( y \), it follows that \( v \) is constant.

The same argument works if \( \text{Im}(f) \) is constant. Alternatively, if \( \text{Im}(f) \) is constant, then \( \text{Re}(if) \) is constant and so \( if \), and thus \( f \), is constant.

Now suppose \( |f| \) is constant. Writing \( f(z) = u(x, y) + iv(x, y) \) for \( z = x + iy \), we then have that \( u(x, y)^2 + v(x, y)^2 \) is constant. In particular, this implies that \( \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial x} \) and that \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y} \) Thus we can again use the Cauchy-Riemann equations:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}
\]

and

\[
\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}
\]

So we get \( \frac{\partial u}{\partial x} \) is equal to both \( \frac{\partial v}{\partial x} \) and \( -\frac{\partial v}{\partial x} \), showing that both are equal to zero, and by the same logic as before, \( f \) is constant.