Solution of problem 2.5.46: First we observe that $\int \int \chi_D(x, y)\,d\mu\,d\nu = 0$ since $\chi_D(x, y)$ is nonzero only when $x = y$ i.e. on the set $y$, which has Lebesgue measure zero. Next note that $\int \int \chi_D(x, y)\,d\mu = 1$ since, as before, $\chi_D$ is only nonzero on the set $x$ and $\nu(x) = 1$, so the integral becomes $\int_{[0, 1]} 1\,d\mu$, which is 1.

For $\int \chi_D\,d(\mu \times \nu)$, first recall that $\mu \times \nu$ was defined to be the measure resulting from the construction done in Caratheodorys Theorem and Theorem 1.14 (page 36), so we have.

$$\mu \times \nu(D) = \inf \left\{ \sum_{i=1}^{+\infty} \mu(A_i)\nu(B_i) : A_i, B_i \in \mathcal{B}_{[0, 1]}, D \subseteq \bigcup_{i=1}^{+\infty} (A_i \times B_i) \right\}, \quad (1)$$

Notice that, for $A_i, B_i \in \mathcal{B}_{[0, 1]}$ with $D \subseteq \bigcup_{i=1}^{+\infty} A_i \times B_i$ we have $(x, x) \in \bigcup_{i=1}^{+\infty} A_i \times B_i$ for each $x \in [0, 1]$. Then $x \in \bigcup_{i=1}^{+\infty} A_i \cap B_i$, so $[0, 1] \subseteq \bigcup_{i=1}^{+\infty} A_i \cap B_i$.

By sub-additivity and monotonicity,

$$1 \leq \mu \left( \bigcup_{i=1}^{+\infty} A_i \cap B_i \right) \leq \sum_{i=1}^{+\infty} \mu(A_i \cap B_i), \quad (2)$$

so for some $k$ we have $\mu(A_k \cap B_k) > 0$ and so $\mu(A_k) > 0$ and $\mu(B_k) > 0$. This implies that $B_k$ has infinite cardinality, so $\nu(B_k) = \infty$ and, therefore:

$$\mu \times \nu(A_k \times B_k) = \mu(A_k)\nu(B_k) = \infty. \quad (3)$$

Thus the sum, $\sum_{i=1}^{+\infty} \mu(A_i)\nu(B_i)$ is infinite and, since $A_i, B_i$ were arbitrary, $\mu \times \nu(D) = +\infty$. Finally:

$$\int \chi_D\,d(\mu \times \nu) = \mu \times \nu(D) = +\infty \quad (4)$$

Solution of problem 2.5.48: for $\int |f|\,d(\mu \times \nu)$, notice that the value of the integral is the measure of the set $\{(n, n), (n + 1, n) : n \in \mathbb{N}\}$ on which $|f| = 1$ and outside of which $|f| = 0$. Since $\nu(\{(n, n), (n + 1, n) : n \in \mathbb{N}\}) = +\infty$, we have $\int |f|\,d(\mu \times \nu) = +\infty$. Interpreting the integral as a sum, we find:

$$\int \int f\,d\mu\,d\nu = \sum_{n=0}^{+\infty} \left( \sum_{m=0}^{+\infty} f(m, n) \right) = \sum_{n=0}^{+\infty} (1 - 1) = 0 \quad (5)$$

and

$$\int \int f\,d\nu\,d\mu = \sum_{m=0}^{+\infty} \left( \sum_{n=0}^{+\infty} f(m, n) \right) = 1 + \sum_{m=1}^{+\infty} (1 - 1) = 1. \quad (6)$$
Solution of problem 2.5.49: by Proposition 2.12 in the book, there is an $\mathcal{M} \otimes \mathcal{M}$-measurable function $g$ such that $g = f \lambda$-a.e. Let $h = f - g$, so that $h$ is $\lambda$-measurable and $h = 0 \lambda$-a.e. By the standard Fubini-Tonelli theorem applied to $g$, it suffices to prove all claims with $h$ in place of $f$. Since $h = 0 \lambda$-a.e. and $\lambda$ is the completion of the measure $\mu \otimes \nu$, there is a set $E \in \mathcal{M} \otimes \mathcal{M}$ with $\mu \otimes \nu(E) = 0$ such that $\{f \neq 0\}$. If we assume that $E \in \mathcal{M} \otimes \mathcal{M}$ and $\mu \otimes \nu(E) = 0$ imply $\mu(E') = \nu(E') = 0$ for a.e. $x$ and $y$, it follows that $h_x = h(x, \cdot) = 0 \nu$-a.e. for $\mu$-a.e. $x$ and that $h^y = h(\cdot, y) = 0 \mu$-a.e. for $\nu$-a.e. $y$. Since $\mu$ and $\nu$ are complete, this implies that $h_x$ and $h^y$ are measurable for $\mu$-a.e. $x$ and $\nu$-a.e. $y$. Instead, if we assume that $h \mathcal{L}$-measurable and $h = 0 \lambda$-a.e. imply $h_x$ and $h^y$ integrable for a.e. $x$ and $y$ with $\int h_x d\nu = \int h^y d\mu = 0$ for a.e. $x$ and $y$, the proof is evidently finished. Indeed, the a.e. defined functions $x \mapsto \int h_x d\nu$ and $y \mapsto \int h^y d\mu = 0$ are 0 a.e. and are, hence, measurable. Furthermore, by using again the completeness of $\mu$ and $\nu$, integrable with integrals 0 (matching the integral of $h$ with respect to $\lambda$). The first claim is immediate from Tonelli’s theorem applied to the characteristic function of $E$, since a nonnegative function with integral 0 must be 0 a.e. The second claim is immediate from the paragraph between the first and second claims (applied in some parts to $|h|$ instead of $h$) and the fact that the integral of a measurable function which is 0 a.e. is 0.

Solution of problem 2.5.53: by Theorem 2.10 we can find a sequence of simple functions $\phi_j \to f$ pointwise and $|\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$. Now, $|\phi_j - f| \to 0$ pointwise and
\[
|\phi_j - f| \leq |\phi_j| + |f| \leq 2|f|.
\] (7)

Applying the Dominated Convergence Theorem,
\[
\lim_{j \to +\infty} \int |\phi_j - f| \, dx = 0.
\] (8)

So, we can find a simple function $\phi = \sum_{j=1}^n a_j \chi_{E_j}$ within $L^1$-distance of $\epsilon/2$ from $f$, that is:
\[
\int |\phi_j - f| \, dx < \epsilon/2.
\] (9)

We can assume, without loss of generality, that the $E_j$’s are disjoint and that for all $j$’s we have $a_j \neq 0$. Now, since:
\[
\sum_{j=1}^n |a_j| \mu(E_j) \int |\phi| \leq \int |f| + \int |\phi - f| < \int |f| + \epsilon/2,
\] (10)

we have that all of the $E_j$’s have finite measure. So, by Theorem 2.40 (item c), we have that, for each $E_j$, there is a finite union of rectangles $F_j$ whose sides are intervals such that $\mu(E_j \triangle F_j) < \epsilon/2 |a_j| n$. Now:
\[
\int \left| \sum_{j=1}^n a_j \chi_{E_j} - \sum_{j=1}^n a_j \chi_{F_j} \right| \leq \sum_{j=1}^n |a_j| \int |\chi_{E_j} - \chi_{F_j}| = \sum_{j=1}^n |a_j| \mu(E_j \triangle F_j) \leq
\]
\[ \sum_{j=1}^{n} \epsilon/2 = \epsilon/2. \quad (11) \]

It follows that:

\[
\int |f - \sum_{j=1}^{n} a_j \chi_{F_j}| \leq \int |f - \phi| + \int |\phi - \sum_{j=1}^{n} a_j \chi_{F_j}| < \epsilon/2 + \epsilon/2. \quad (12)
\]

Since for each \( j \) we have that \( F_j \) is a finite union of rectangles whose sides are intervals, we have that

\[
\sum_{j=1}^{n} a_j \chi_{E_j} = \sum_{k=1}^{m} c_k \chi_{R_k} \quad (13)
\]

where, for each \( k \), \( R_k \) is a rectangle whose sides are intervals, \( c_k \neq 0 \). Take \( \psi = \sum_{k=1}^{m} c_k \chi_{R_k} \). So we proved that, for any \( \epsilon > 0 \), there is a simple function \( \psi = \sum_{k=1}^{m} c_k \chi_{R_k} \) such that, for each \( k \), \( R_k \) is a rectangle whose sides are intervals, \( c_k \neq 0 \) and

\[
\int |\psi - f| < \epsilon. \quad (14)
\]

Now, for each \( k \), let \( \phi_k \) be a continuous function such that \( |\chi_{(R_k - \phi_k)}| < \epsilon/2 |c_k| m \) (such function always exists). Now, if we set \( \psi = \sum_{k=1}^{m} c_k \phi_k \), it’s clear that \( g \) is continuous and we have:

\[
< \epsilon/2 + \int \left| \sum_{k=1}^{m} c_k \chi_{R_k} - \sum_{k=1}^{m} c_k \phi_k \right| \leq \epsilon/2 + \sum_{k=1}^{m} |c_k| \int |\chi_{R_k} - \phi_k| \leq \epsilon/2 + \sum_{k=1}^{m} |c_k| \epsilon/2 |c_k| m \quad = \epsilon/2 + \epsilon/2. \quad (15)
\]

**Solution of problem 2.5.54:** (a) if \( f \) is Borel measurable on \( \mathbb{R}^n \), then so is \( f \circ T \). This is true even if \( T \) is not invertible, since, since the definition of a measurable function on page 43, a Borel function of a Borel-measurable function is always Borel measurable. (b) Equation (2.45); If \( T \) is not invertible and (for example) \( f = e^{-||x||^2} \), then \( \text{det}(T) = 0 \) and \( (T(x))dx = +\infty \). The equation is false under the convention \( 0 \cdot \infty = 0 \). Furthermore, \( |T(E)| = |\text{det}(T)||E| \) is true, in a sense, if \( T \) is not invertible because the range of \( T \) is a lower-dimensional subspace of \( \mathbb{R}^n \) and \( \text{det}(T) = 0 \). Thus \( T(E) \) is a Lebesgue null set even if is not a Borel set.
Solution of problem 2.5.55: the strategy is to first integrate $|f|$. If the integral exists and is finite, then $f \in L^1$ and, by Fubini’s theorem, all three integrals are equal. As to (a), it’s easy to see (by integrating w.r.t. $y$ the sum of two integrals w.r.t. $x$ respectively over the integrals $[0,y]$ and $[y,1]$) that $\int |f| \, dm^2$ doesn’t exist. Indeed we have:

$$
\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \, dy = \int_0^1 \left[ \frac{-x}{x^2 + y^2} \right]_0^1 \, dy = \int_0^1 \frac{-1}{1+y^2} \, dy = \\
= \left[ -\tan^{-1}(y) \right]_0^1 = -\frac{\pi}{4}, \quad (16)
$$

$$
\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \, dy = \int_0^1 \left[ \frac{y}{x^2 + y^2} \right]_0^1 \, dy = \int_0^1 \frac{1}{1+x^2} \, dy = \\
= \left[ \tan^{-1}(x) \right]_0^1 = \frac{\pi}{4}. \quad (17)
$$

The argument is similar for all the other cases.