Homework 5 Math 420/507, 2016: solutions of the exercises.

Solution of problem 2.5.46: First we observe that $\int\int \chi_D(x,y)d\mu\, d\nu = 0$ since $\chi_D(x,y)$ is nonzero only when $x = y$ i.e. on the set y, which has Lebesgue measure zero. Next note that $\int\int \chi_D(x,y)d\mu = 1$ since, as before, χ_D is only nonzero on the set x and $\nu(x) = 1$, so the integral becomes $\int_{[0,1]} 1 d\mu$, which is 1. For $\int \chi_D d(\mu\times\nu)$, first recall that $\mu\times\nu$ was defined to be the measure resulting from the construction done in Caratheodorys Theorem and Theorem 1.14 (pag 36), so we have.

$$\mu\times\nu(D) = \inf\left\{\sum_{i=1}^{\infty}\mu(A_i)\nu(B_i) : A_i, B_i \in \mathcal{B}_{[0,1]}, D \subseteq \bigcup_{i=1}^{+\infty}(A_i\times B_i)\right\}. \tag{1}$$

Notice that, for $A_i, B_i \in \mathcal{B}_{[0,1]}$ with $D \subseteq \bigcup_{i=1}^{+\infty} A_i \times B_i$ we have $(x,x) \in \bigcup_{i=1}^{+\infty} A_i\times B_i$ for each $x \in [0,1]$. Then $x \in \bigcup_{i=1}^{+\infty} A_i\cap B_i$, so $[0,1] \subseteq \bigcup_{i=1}^{+\infty} A_i\cap B_i$. By sub-additivity and monotonicity,

$$1 \leq \mu\left(\bigcup_{i=1}^{+\infty} A_i\cap B_i\right) \leq \sum_{i=1}^{+\infty}\mu(A_i\cap B_i), \tag{2}$$

so for some k we have $\mu(A_k\cap B_k) > 0$ and so $\mu(A_k) > 0$ and $\mu(B_k) > 0$. This implies that B_k has infinite cardinality, so $\nu(B_k) = \infty$ and, therefore:

$$\mu\times\nu(A_k\times B_k) = \mu(A_k)\nu(B_k) = \infty. \tag{3}$$

Thus the sum, $\sum_{i=1}^{+\infty}\mu(A_i)\nu(B_i)$ is infinite and, since A_i, B_i were arbitrary, $\mu\times\nu(D) = +\infty$. Finally:

$$\int \chi_D d(\mu\times\nu) = \mu\times\nu(D) = +\infty \tag{4}$$

Solution of problem 2.5.48: for $\int |f| d(\mu\times\nu)$, notice that the value of the integral is the measure of the set $\{(n,n),(n+1,n) : n \in \mathcal{N}\}$ on which $|f| = 1$ and outside of which $|f| = 0$. Since $\nu(\{(n,n),(n+1,n) : n \in \mathcal{N}\}) = +\infty$, we have $\int |f|\, d(\mu\times\nu) = +\infty$. Interpreting the integral as a sum, we find:

$$\int\int f d\mu d\nu = \sum_{n=0}^{+\infty}\left(\sum_{m=0}^{+\infty} f(m,n)\right) = \sum_{n=0}^{+\infty}(1-1) = 0 \tag{5}$$

and

$$\int\int f d\nu d\mu = \sum_{m=0}^{+\infty}\left(\sum_{n=0}^{+\infty} f(m,n)\right) = 1 + \sum_{m=1}^{+\infty}(1-1) = 1. \tag{6}$$

Solution of problem 2.5.49: by Proposition 2.12 in the book, there is an $\mathcal{M}\otimes\mathcal{M}$-measurable function g such that $g = f\lambda$-a.e. Let $h = f - g$, so that h is λ-measurable and $h = 0\lambda$-a.e. By the standard Fubini-Tonelli theorem applied to g, it suffices to prove all claims with h in place of f. Since $h = 0\lambda$-a.e. and λ is the completion of the measure $\mu\otimes\nu$, there is a set $E \in \mathcal{M}\otimes\mathcal{M}$ with $\mu\otimes\nu(E) = 0$ such that $\{f \neq 0\}$. If we assume that $E \in \mathcal{M}\otimes\mathcal{M}$ and $\mu\otimes\nu(E) = 0$ imply $\mu(E^y) = \nu(E_x) = 0$ for a.e. x and y, it follows that $h_x = h(x,\cdot) = 0$ ν-a.e. for μ-a.e. x and that $h^y = h(\cdot, y) = 0$ μ-a.e. for ν-a.e. y. Since μ and ν are complete, this implies that h_x and h^y are measurable for μ-a.e. x and ν-a.e. y. Instead, if we assume that h $\mathcal{L}$-measurable and $h = 0$ λ-a.e. imply h_x and h^y integrable for a.e. x and y with $\int h_x d\nu = \int h^y d\mu = 0$ for a.e. x and y, the proof is evidently finished. Indeed, the a.e. defined functions $x \to \int h_x d\nu$ and $y \to \int h^y d\mu = 0$ are 0 a.e. and are, hence, measurable. Furthermore, by using again the completeness of μ and ν, integrable with integrals 0 (matching the integral of h with respect to λ). The first claim is immediate from Tonelli's theorem applied to the characteristic function of E, since a nonnegative function with integral 0 must be 0 a.e. The second claim is immediate from the paragraph between the first and second claims (applied in some parts to $|h|$ instead of h) and the fact that the integral of a measurable function which is 0 a.e. is 0.

Solution of problem 2.5.53: by Theorem 2.10 we can find a sequence of simple functions $\phi_j \to f$pointwise and $|\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$. Now, $|\phi_j - f| \to 0$ pointwise and

$$|\phi_j - f| \;\leq\; |\phi_j| + |f| \;\leq\; 2\,|f|\,. \tag{7}$$

Applying the Dominated Convergence Theorem,

$$\lim_{j\to+\infty} |\phi_j - f|\,dx \;=\; 0. \tag{8}$$

So, we can find a simple function $\phi = \sum_{j=1}^{n} a_j\chi_{E_j}$ within L^1-distance of $\epsilon/2$ from f, that is:

$$\int |\phi_j - f|\,dx \;<\; \epsilon/2. \tag{9}$$

We can assume, without loss of generality, that the E_j's are disjoint and that for all j's we have $a_j \neq 0$. Now, since:

$$\sum_{j=1}^{n} |a_j|\,\mu(E_j) \;\int |\phi| \;\leq\; \int |f| + \int |\phi - f| \;<\; \int |f| + \epsilon/2, \tag{10}$$

we have that all of the E_j's have finite measure. So, by Theorem 2.40 (item c), we have that, for each E_j, there is a finite union of rectangles F_j whose sides are intervals such that $\mu(E_j \triangle F_j) < \epsilon/2\,|a_j|\,n$. Now:

$$\int \left| \sum_{j=1}^{n} a_j\chi_{E_j} - \sum_{j=1}^{n} a_j\chi_{F_j} \right| \;\leq\; \sum_{j=1}^{n} |a_j| \int \left|\chi_{E_j} - \chi_{F_j}\right| = = \;\sum_{j=1}^{n} |a_j|\,\mu(E_j \triangle F_j) \;\leq$$

$$\leq \sum_{j=1}^{n} \epsilon/2n \;=\; \epsilon/2. \quad (11)$$

It follows that:

$$\int \left| f - \sum_{j=1}^{n} a_j \chi_{F_j} \right| \;\leq\; \int |f-\phi| + \int \left| \phi - \sum_{j=1}^{n} a_j \chi_{F_j} \right| \;<\; \epsilon/2 + \epsilon/2. \qquad (12)$$

Since for each j we have that F_j is a finite union of rectangles whose sides are intervals, we have that

$$\sum_{j=1}^{n} a_j \chi_{E_j} \;=\; \sum_{k=1}^{m} c_k \chi_{R_k} \qquad (13)$$

where, for each k, R_k is a rectangle whose sides are intervals, $c_k \neq 0$. Take $\psi = \sum_{k=1}^{m} c_k \chi_{R_k}$. So we proved that, for any $\epsilon > 0$, there is a simple function $\psi = \sum_{k=1}^{m} c_k \chi_{R_k}$ such that, for each k, R_k is a rectangle whose sides are intervals, $c_k \neq 0$ and

$$\int |\psi - f| \;<\; \epsilon. \qquad (14)$$

Now, for each k, let ϕ_k be a continuous function such that $|\chi(R_k - \phi_k)| < \epsilon/2\,|c_k|\,m$(such function always exists). Now, if we set $\psi = \sum_{k=1}^{m} c_k \phi_k$, it's clear that g is continuous and we have:

$$\int |f-g| \;\; \int |f-\psi| + \int |\psi - g| \;<$$

$$<\; \epsilon/2 + \int \left| \sum_{k=1}^{m} c_k \chi_{R_k} - \sum_{k=1}^{m} c_k \phi_k \right\| \leq \epsilon/2 + \sum_{k=1}^{m} |c_k| \int |\chi_{R_k} - \phi_k| \;\leq$$

$$\leq \epsilon/2 + \sum_{k=1}^{m} |c_k|\,\epsilon/2\,|c_k|\,m \;=\; \epsilon/2 + \epsilon/2. \qquad (15)$$

Solution of problem 2.5.54: (a) if f is Borel measurable on $\mathbb{R}^n$, then so is $f \circ T$. This is true even if T is not invertible, since, since by the definition of a measurable function on page 43, a Borel function of a Borel-measurable function is always Borel measurable. (b) Equation (2.45); If T is not invertible and (for example) $f = e^{-|\cdot|^2}$, then $det(T) = 0$ and $(T(x))dx = +\infty$. The equation is false under the convention $0 \cdot \infty = 0$. Furthermore, $|T(E)| = |det(T)|\,|E|$ is true, in a sense, if T is not invertible because the range of T is a lower-dimensional subspace of $\mathbb{R}^n$ and $det(T) = 0$. Thus $T(E)$ is a Lebesgue mull set even if is not a Borel set.

Solution of problem 2.5.55: the strategy is to first integrate $|f|$. If the integral exists and is finite, then $f \in L^1$ and, by Fubini's theorem, all three integrals are equal. As to (a), it's easy to see (by integrating w.r.t. y the sum of two integrals w.r.t. x respectively over the integrals $[0, y]$ and $[y, 1]$) that $\int |f|\, dm^2$ doesn't exist. Indeed we have:

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dxdy = \int_0^1 \left[\frac{-x}{x^2 + y^2}\right]_0^1 dy = \int_0^1 \frac{-1}{1 + y^2} dy = \\ = \left[-\tan^{-1}(y)\right]_0^1 = -\frac{\pi}{4}, \qquad (16)$$

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dxdy = \int_0^1 \left[\frac{y}{x^2 + y^2}\right]_0^1 dy = \int_0^1 \frac{1}{1 + x^2} dy = \\ = \left[\tan^{-1}(x)\right]_0^1 = \frac{\pi}{4}. \qquad (17)$$

The argument is similar for all the other cases.