Homework 3 Math 420/507, 2016: solutions of the exercises.

Solution of problem 1.4.18: as to **a**, by the definition of outer measure we know that:

$$\mu^*(E) \;=\; \inf\left\{\sum_{j=1}^{+\infty}\mu_0(A_j) : A_j \in \mathcal{A}, E \subset \cup_{j=1}^{+\infty} A_j\right\}. \tag{1}$$

Let $A = \cup_{j=1}^{+\infty} A_j$ as above. Then $A \in \mathcal{A}_\sigma$ and $E \in \mathcal{A}$. For each j we can construct a sequence $B_j^k{}_{k=1}^{+\infty} \subset A$ such that $A_j \subset \cup_{j,k=1}^{+\infty} B_j^k$. It follows that, since

$$\mu^*(A_j) \;=\; \inf\left\{\sum_{j=1}^{+\infty}\mu_0(B_j^k) : B_j^k \in \mathcal{A}, a_j \subset \cup_{j,k=1}^{+\infty} B_j^k\right\}, \tag{2}$$

we have that

$$\mu^*(A_j) \;\leq\; \mu_0(A_j) + \epsilon 2^{-j}, \tag{3}$$

for all j and $\epsilon > 0$. Thus:

$$\mu^*(A) \;\leq\; \sum_{j=1}^{+\infty}\mu^*(A_j) \;\leq\; \sum_{j=1}^{+\infty}(\mu_0(A_j) + \epsilon 2^{-j}) \;=\; \mu^*(E) + \epsilon. \tag{4}$$

Since ϵ is arbitrary, we are done.

As to **b**, we suppose E μ^*-measurable. From **a**, we know that for all $n \in \mathbb{N}$ there exists $B_n \in A_\sigma$ with $E \subset B_n$ and $\mu^*(B_n) - \mu^*(E) \leq 1/n$. Let $B = \cap_{n=1}^{+\infty} B_n \in A_\sigma$. Since E is μ^*-measurable, we have $\mu^*(B_n) = \mu^*(B_n \cap E) + \mu^*(B_n \cap E^c)$, hence: $\mu^*(B \cap E^c) \leq \mu^*(B_n \cap E^c) = \mu^*(B_n) - \mu^*(E) \leq 1/n$ for all $n \in \mathbb{N}$. Hence we have $\mu^*(B\backslash E) = 0$. To show the converse, let's suppose $B \in A_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B\backslash E) = 0$. From part **a**, we know that for all $n \in \mathbb{N}$ there exists $A_n A_\sigma$ with $B\backslash E \subset A_n$ and $\mu^*(A_n) - \mu^*(B\backslash E) \leq 1/n$. But, since $\mu^*(B\backslash E) = 0$, then $\mu^*(A_n) \leq 1_n$. Let's set $A = \cap_{n=1}^{+\infty} A_n$. Then, A is μ^*-measurable (since $A_{\sigma\delta}$ and the set of all μ^*-measurable sets is a σ-algebra) such that $B\backslash E \subset A$ and $\mu^*(A) = 0$. By Carathedory's theorem we know that the restriction of $\mu^*$to μ^*-measurable sets is a complete measure. Then $B\backslash E$ is μ^*-measurable. Also, since $BA_{\sigma\delta}$, B is also μ^*-measurable and we get $E = (B^c \cup (B \cap E^c))^c$. Thus E is μ^*-measurable.

As to **c**, let μ_0 be σ-finite and let's set $X = \cup_{i=1}^{+} X_i$ where $X_i M$ and $\mu(X_i) < +\infty$. Now let's suppose E is μ^*-measurable and $\mu^*(E) = +\infty$. Let's set $E_n =$

$(E \cap \cup_{i=1}^{n} X_i)$. Then $\mu^*(E_n) < +\infty$ and $E = \cup_{n=1}^{+\infty} E_n$. Let $\epsilon > 0$; from part **a** and for all $n \in \mathbb{N}$ we have that there exists a set $C_n A_\sigma$ such that $E_n \subset C_n$ and:

$$\mu^*(C_n \backslash E_n) \ = \mu^*(C_n) - \mu^*(E_n) \ \leq \ \epsilon/2n. \tag{5}$$

We set:

$$C_\epsilon \ = \ \cup_{n=1}^{+\infty} C_n. \tag{6}$$

We have:

$$\mu^*(C_\epsilon \backslash E) \ = \ \mu^*(\cup_{n=1}^{+\infty} C_n \backslash E) \ = \ \mu^*(\cup_{n=1}^{+\infty} C_n \backslash E_n) \ \leq$$

$$\leq \sum_{n=1}^{+\infty} \mu^*(C_n \backslash E_n) \ \leq \ \epsilon. \tag{7}$$

We have that $C := \cap_{n=1}^{+\infty} C_{1/n} \in A_{\sigma\delta}$ implies $\mu^*(C \backslash E) = 0$. Indeed $\mu^*(C \backslash E) \leq \mu^*(C_{1/n} \backslash E) \leq 1/n \to 0$.

Solution of problem 1.4.21: assume E is a locally measurable set. It suffices to show that

$$\mu^*(F) \ \geq \ \mu^*(F \cap E) + \mu^*(F \cap E^c) \tag{8}$$

for all $F \subset X$ with $\mu^*(F) < +\infty$. So let $F \subset X$ be such and let $\epsilon > 0$. Using exercise 18, we find a set $A \in A_\sigma$ ($A_\sigma \subset \mathcal{M}$ with $F \subset A$ and $\mu^*(A) \leq \mu^*(F) + \epsilon$ (thus $\mu^*(A) < +\infty$). We have that $E \cap A \in \mathcal{M}$, since E is locally measurable and $A \in \mathcal{M}$ is such that $\mu^*(A) < +\infty$. It follows that

$$\mu^*(A) \ = \ \mu^*(A \cap (E \cap A)) + \mu^*(A \cap (E \cap A)^c) \ = \ \mu^*(A \cap E) + \mu^*(A \cap E^c). \tag{9}$$

Since $F \subset A$:

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \ \geq \ \mu^*(F \cap E) + \mu^*(F \cap E^c). \tag{10}$$

Therefore

$$\mu^*(F) + \epsilon \ \geq \ \mu^*(A) \ \geq \ \mu^*(F \cap E) + \mu^*(F \cap E^c). \tag{11}$$

The thesis follows because ϵ is arbitrary.

Solution of problem 1.4.22: we show that $\mathcal{M}^* = \overline{\mathcal{M}}$. Then, the uniqueness follows from Theorem 1.9 and Carathodory's Theorem (which guarantees that μ^* is complete). Let μ^* be the outer measure induced by μ, $\mathcal{M}^*$ the μ^*-measurable sets and $\overline{\mu}$ the measure μ^* restricted to $\mathcal{M}^*$. We prove that $\overline{\mu}$ on $\mathcal{M}^*$ is the completion of the measure μ defined on $\mathcal{M}$. We first prove that if $(X, \mathcal{F}, \nu)$ is a measure space, and let $\overline{\mathcal{F}}$ is the completion of $\mathcal{F}$ according to the definition:

$$\overline{\mathcal{F}} \ \{E \cup F : E \in \mathcal{F}, F \subset N forsomeN \in \mathcal{F} suchthat \nu(N) = 0\}. \tag{12}$$

Then,

$$\overline{\mathcal{F}} \ \{G \backslash D : G \in \mathcal{F}, D \subset N forsomeN \in \mathcal{F} suchthat \nu(N) = 0\}. \tag{13}$$

Indeed, let's take $E \cup F \in \overline{\mathcal{F}}$, where $E \in \mathcal{F}$ and $F \subset N$ for some $N \in \overline{\mathcal{F}}$ such that $\nu(N) = 0$. Then:

$$E \cup F \ = \ E \cup N(N(F \cup E)). \tag{14}$$

Note that $E \cup N \in \mathcal{F}$ and $N \backslash (F) \subset N$. Conversely, consider GD where $G \in \mathcal{F}$and there is $N \in \mathcal{F}$ with $\nu(N) = 0$ such that $D \subset N$. Then:

$$GD \ = \ (GN) \cup (N \cap G)D). \tag{15}$$

But $GN \in \mathcal{F}$ and $(N \cap G)D \subset N$. So we are done. Now we remember that:

$$\overline{\mathcal{M}} \ \{G \cup F : G \in \mathcal{M}, F \subset N forsomeN \in \mathcal{M} suchthat \mu(N) = 0\}. \tag{16}$$

and that, for such a set $G \cup F$ as in the definition, $\overline{\mu}(G \cup F) := \mu(G)$. We want to prove that $\overline{\mathcal{M}} = \mathcal{M}^*$ and $\mu^*(E) = \overline{\mu}(E)$ for $E \in \mathcal{M}^{ast}$. We know from Exercise 18 that, if E is μ^*-measurable, there exists B in $\mathcal{M}_{\sigma\delta}$ such that $E \subset B$ and $\mu^*(B) = 0$. However, because $\mathcal{M}$ is a σ-algebra, $\mathcal{M}_{\sigma\delta} = \mathcal{M}$ and $B \in \mathcal{M}$. By similar reasoning, since $B \backslash E$ is μ^*-measurable, there exists $C \in \mathcal{M}$ such that $B \backslash E \subset C$ and $\mu^*(C) = \mu(C) = 0$. Because $E = B \backslash (B \backslash E)$, where $B \in \mathcal{M}$, our first argument tells us that $E \in \overline{\mathcal{M}}$. This establishes that $\mathcal{M}^* \subseteq \overline{\mathcal{M}}$. Instead, $\overline{\mathcal{M}} \subseteq \mathcal{M}^*$ is true because any set of outer measure 0 is in$\mathcal{M}^*$. So if $G \in \mathcal{M} \subset \mathcal{M}^*$ and $F \subset N$ with $N \in \mathcal{M}$ and $\mu(N) = 0$, we fave $F \in \mathcal{M}^*$ as well and, thus, $G \cup F \in \mathcal{M}^*$. Finally, if $G \in \mathcal{M}$ and $F \subset N$, where $N \in \mathcal{M}$ and $\mu(N) = 0$, then we get:

$$\mu(G) \ = \ \mu^*(G) \ \geq \ \mu^*(G \cup F) \ \leq \ \mu^*(G) + \mu^*(F) \ = \mu^*(G) \ = \ \mu(G). \tag{17}$$

Thus $\mu^*(G \cup F) = \mu(G) = \overline{\mu}(G)$.

Solution of problem 1.5.31: following the hint, the problem is easy if $|E| < +\infty$. In the general case, we have $\mathbb{R} = \cup_{n \in mathbbZ}(n, n+1]$, therefore,

since $|E| > 0$, it must be the case that $|E?(n, n+1]| > 0$ for some n. But $|E?(n, n+1]| \leq 1$, so the problem may be effectively reduced to the previous case.

Solution of problem 1.5.33: Consider a Cantor-type set of positive measure over $[0, 1]$, then consider the countable family of intervals that you have to remove to construct it. For each of these intervals (but we can work with their closures), we can construct other Cantor-type sets, but proportioned to the length of the intervals. And so on. The set that we get after this construction is a Borel set and it's easy to see that satisfies the condition we want.