Homework 1 Math 420507, 2016: solutions of the exercises.

Solution of problem 1.2.3, pag. 24. As to **a**, there exists $A_1 \in \mathcal{M}$ such that the restriction of the collection of all intersections of the sets of $\mathcal{M}$ with A_1 (which is a σ algebra; let's denote it by $A_1^c \cap \mathcal{M}$) is still infinite. Indeed, if no such set exists, then we can choose a set $A \in \mathcal{M}$ with $A \neq \emptyset$ and $A^c \cap \mathcal{M}$ finite. But even $A \cap \mathcal{M}$ is finite (otherwise it could have been $A_1 = A^c$). Therefore $\mathcal{M}$ is finite, which is absurd. Now we consider the σ algebra $A_1^c \cap \mathcal{M}$ and prove, by the same argument, the existence of a set $A_2 \in A_1^c \cap \mathcal{M}$ such that $A_2^c \cap (A_1^c \cap \mathcal{M})$ is infinite. Then we have just to iterate the previous argument.

As to **b**, if $(X, \mathcal{M})$ is not finite and we suppose that $\mathcal{M} = \{E_k\}_{k\in\mathbb{N}}$, for any $x \in X$ we have that the set:

$$E_x := \bigcap_{E_k \supset x} E_k \tag{1}$$

is still in $\mathcal{M}$. It's easy to understand that E_x is the smallest set of $\mathcal{M}$ to contain x and the collection of all E_x's is a partition of X. Indeed, let's assume $E_x \cap E_y \neq \emptyset$. If $x \notin E_y$, then $E_x \backslash E_y \in \mathcal{M}$ and is smaller than E_x, which is a contraddiction with what E_x is. Therefore, $x \in E_y$ and, by the same argument, $y \in E_x$. But then, by the definition of E_x and E_y, we get$E_x \subseteq E_y$ and $E_x \supseteq E_y$, that is $E_x = E_y$. Now, each set E in$\mathcal{M}$ can be written as the union of such sets:

$$E_k := \bigcup_{x\in E} E_x. \tag{2}$$

Therefore, the partition of E_x's cannot be finite, otherwise $\mathcal{M}$ would be finite too. As consequence, we can form all the sets in $\mathcal{M}$ by taking all the possible (disjoint) unions of sets of the partition. This means that card$(\mathcal{M})$ is equal to card$(\mathcal{P}(\mathcal{M}))$. But the latter is uncountable and this is a contradiction.

Solution of problem 1.2.4, pag. 24 if $\mathcal{A}$ is a σ-algebra, then it's an algebra and it's closed under countable increasing unions. On the other hand, if $\mathcal{A}$ is an algebra and it's closed under countable increasing unions, let A_k be a sequence of sets in $\mathcal{A}$. We have to show that $\cup_{\mathbb{N}} A_k$ is in $\mathcal{A}$. But if we set $B_N := \cup_{k=1}^{N} A_k$, we have that $B_N \subset B_{N+1}$, $B_N \in \mathcal{A}$ for all N and:

$$\cup_{k=1}^{\infty} B_N \;=\; \cup_{k=1}^{+\infty} A_k. \tag{3}$$

Therefore $\cup_{k=1}^{+\infty} A_k \in \mathcal{A}$ and $\mathcal{A}$ is a σ-algebra.

Solution of problem 1.2.5, pag. 24 By definition, $\mathcal{M}$ is the smallest σ-algebra generated by $\mathcal{E}$. Let $\mathcal{P}_{\mathbb{N}}(\mathcal{E})$ the collection of all countable subfamilies of $\mathcal{E}$. Then we have to show that:

$$\sigma(\mathcal{E}) \;=\; \bigcup_{\mathcal{N}\in\mathcal{P}_{\mathbb{N}}(\mathcal{E})} \sigma(\mathcal{N}). \tag{4}$$

We have that if $A_k \in \cup_{\mathcal{N}\in\mathcal{P}_{\mathbb{N}}(\mathcal{E})}\sigma(\mathcal{N})$ for all k, then A_k must belong to some $\sigma(\mathcal{N})$, say $\sigma(\mathcal{N}_{\|})$. then we have:

$$\bigcup_{k\in\mathbb{N}} A_k \subset \bigcup_{k\in\mathbb{N}} \sigma(\mathcal{N}) \subset \bigcup_{\mathcal{N}\in\mathcal{P}_{\mathbb{N}}(\mathcal{E})} \sigma(\mathcal{N}). \tag{5}$$

Moreover, if if $A \in \cup_{\mathcal{N}\in\mathcal{P}_{\mathbb{N}}(\mathcal{E})}\sigma(\mathcal{N})$, A must belong to some $\sigma(\mathcal{N})$, say $\sigma(\mathcal{N}_{\prime})$. Therefore $A^c \in \sigma(\mathcal{N}_{\prime})$. This shows that $\bigcup_{\mathcal{N}\in\mathcal{P}_{\mathbb{N}}(\mathcal{E})}\sigma(\mathcal{N})$ is a σ-algebra. Now we show that

$$\sigma(\mathcal{E}) \;=\; \bigcup_{\mathcal{N}\in\mathcal{P}_{\mathbb{N}}(\mathcal{E})} \sigma(\mathcal{N}). \tag{6}$$

If $A \in \mathcal{M} = \sigma(\mathcal{E})$, A must belong to some $\mathcal{N} \in \mathcal{P}_{\mathbb{N}}(\mathcal{E})$, say $\mathcal{N}$. But, by the definition of $\sigma(\mathcal{E})$, we have :

$$\sigma(\mathcal{E}) \;\subseteq= \bigcup_{\mathcal{N}\in\mathcal{P}_{\mathbb{N}}(\mathcal{E})} \sigma(\mathcal{N}). \tag{7}$$

On the other hand, if $\mathcal{N} \in \mathcal{P}_{\mathbb{N}}(\mathcal{E})$, then $\sigma(\mathcal{N}) \subseteq \sigma(\mathcal{E})$ and

$$\bigcup_{\mathcal{N}\in\mathcal{P}_{\mathbb{N}}(\mathcal{E})} \sigma(\mathcal{N}) \;\subseteq\; \sigma(\mathcal{E}) \tag{8}$$

Solution of problem 1.3.11, pag. 27 If μ is a measure, then the rest is obvious (see Theo. 1.8). Conversely, let μ be a f.a.m. that is continuous from below. If A_k is a seq. of disjoint μ-measurable sets, we have:

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \;=\; \mu\left(\bigcup_{N=1}^{+\infty}\bigcup_{k=1}^{N} A_k\right). \tag{9}$$

Now we use the property of continuity from below:

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \;=\; \mu\left(\bigcup_{N=1}^{+\infty}\bigcup_{k=1}^{N} A_k\right) \;=\; \lim_{N\to+\infty}\mu\left(\bigcup_{k=1}^{N} A_k\right) \;=$$

$$= \lim_{N\to+\infty}\sum_{k=1}^{N}\mu(A_k) \;=\; \sum_{k=1}^{+\infty}\mu(A_k)\,. \tag{10}$$

Therefore μ is a measure.

Now, again, if If μ is a measure, then the rest is obvious (see Theo. 1.8). Conversely, we suppose $\mu(X) < +\infty$ and let μ be a f.a.m. that is continuous from above. Then, If A_k is a seq. of disjoint μ-measurable sets, we have:

$$\bigcup_{k=1}^{\infty}A_k \;=\; X\backslash\bigcap_{k=1}^{\infty}A_k^c \;=\; X\backslash\bigcap_{N=1}^{\infty}\bigcap_{k=1}^{N}A_k^c. \tag{11}$$

But $\bigcap_{k=1}^{N}A_k^c$ is a decreasing seq., therefore:

$$\mu(X)-\mu\left(\bigcap_{N=1}^{\infty}\bigcap_{k=1}^{N}A_k^c\right) \;=\; \mu(X)-\lim_{N\to+\infty}\mu\left(\bigcap_{k=1}^{N}A_k^c\right) \;=\; \mu(X)-\lim_{N\to+\infty}\mu\left(X\backslash\bigcup_{k=1}^{N}A_k^c\right) \;=$$

$$= \lim_{N\to+\infty}\mu\left(\bigcup_{k=1}^{N}A_k^c\right)\;0\;\lim_{N\to+\infty}\sum_{k=1}^{N}\mu(A_k) \;=\; \sum_{k=1}^{+\infty}\mu(A_k)\,. \tag{12}$$

Therefore μ is a measure.

Solution of problem 1.3.11, pag. 27 By hypothesis, there exists a seq. E_k of μ-measurable sets such that $\cup_{k=1}^{\infty}E_k = X$, $\mu(E_k) < +\infty$. It's easy to see that we can suppose this seq. to be increasing. Now, if E is a μ-measurable set with $\mu(E) = +\infty$, let' consider the seq. $E_k \cap E$. We have:

$$\bigcup_{k=1}^{+\infty}E\cap E_k\;0 = \;E\;\Rightarrow\;+\infty = \lim_{N\to+\infty}\mu(E\cap E_k). \tag{13}$$

Therefore ther must be an index k_0 such that $\mu(E\cap E_{k_0}) > 0$. But $\mu(E\cap E_{k_0}) \le \mu(E_{k_0}) < +\infty$. Therefore μ is semi-finite.