Solution of problem 1.2.3, pag. 24. As to a, there exists $A_1 \in \mathcal{M}$ such that the restriction of the collection of all intersections of the sets of $\mathcal{M}$ with $A_1$ (which is a $\sigma$-algebra; let’s denote it by $A_1^c \cap \mathcal{M}$) is still infinite. Indeed, if no such set exists, then we can choose a set $A \in \mathcal{M}$ with $A \neq \emptyset$ and $A^c \cap \mathcal{M}$ finite. But even $A \cap \mathcal{M}$ is finite (otherwise it could have been $A_1 = A^c$). Therefore $\mathcal{M}$ is finite, which is absurd. Now we consider the $\sigma$-algebra $A_1^c \cap \mathcal{M}$ and prove, by the same argument, the existence of a set $A_2 \in A_1^c \cap \mathcal{M}$ such that $A_2^c \cap (A_1^c \cap \mathcal{M})$ is infinite. Then we have just to iterate the previous argument.

As to b, if $(X, \mathcal{M})$ is not finite and we suppose that $\mathcal{M} = \{ E_k \}_{k \in \mathbb{N}}$, for any $x \in X$ we have that the set:

$$E_x := \bigcap_{E_k \supseteq x} E_k$$

is still in $\mathcal{M}$. It’s easy to understand that $E_x$ is the smallest set of $\mathcal{M}$ to contain $x$ and the collection of all $E_x$’s is a partition of $X$. Indeed, let’s assume $E_x \cap E_y \neq \emptyset$. If $x \notin E_y$, then $E_x \setminus E_y \in \mathcal{M}$ and is smaller than $E_x$, which is a contradiction with what $E_x$ is. Therefore, $x \in E_y$ and, by the same argument, $y \in E_x$. But then, by the definition of $E_x$ and $E_y$, we get $E_x \subseteq E_y$ and $E_y \supseteq E_x$; that is $E_x = E_y$. Now, each set $E$ in $\mathcal{M}$ can be written as the union of such sets:

$$E_k := \bigcup_{x \in E} E_x.$$  \hspace{1cm} (2)

Therefore, the partition of $E_x$’s cannot be finite, otherwise $\mathcal{M}$ would be finite too. As consequence, we can form all the sets in $\mathcal{M}$ by taking all the possible (disjoint) unions of sets of the partition. This means that $\text{card} (\mathcal{M})$ is equal to $\text{card} (\mathcal{P}(\mathcal{M}))$. But the latter is uncountable and this is a contradiction.

Solution of problem 1.2.4, pag. 24 if $\mathcal{A}$ is a $\sigma$-algebra, then it’s an algebra and it’s closed under countable increasing unions. On the other hand, if $\mathcal{A}$ is an algebra and it’s closed under countable increasing unions, let $A_k$ be a sequence of sets in $\mathcal{A}$. We have to show that $\cup_{k=1}^{N} A_k$ is in $\mathcal{A}$. But if we set $B_N := \cup_{k=1}^{N} A_k$, we have that $B_N \subset B_{N+1}$, $B_N \in \mathcal{A}$ for all $N$ and:

$$\cup_{k=1}^{\infty} B_N = \cup_{k=1}^{+\infty} A_k.$$ \hspace{1cm} (3)

Therefore $\cup_{k=1}^{+\infty} A_k \in \mathcal{A}$ and $\mathcal{A}$ is a $\sigma$-algebra.
Solution of problem 1.2.5, pag. 24 By definition, \( M \) is the smallest \( \sigma \)-algebra generated by \( E \). Let \( \mathcal{P}_\mathcal{N}(E) \) the collection of all countable subfamilies of \( E \). Then we have to show that:

\[
\sigma(E) = \bigcup_{\mathcal{N} \in \mathcal{P}_\mathcal{N}(E)} \sigma(\mathcal{N}).
\]  

We have that if \( A_k \in \bigcup_{\mathcal{N} \in \mathcal{P}_\mathcal{N}(E)} \sigma(\mathcal{N}) \) for all \( k \), then \( A_k \) must belong to some \( \sigma(\mathcal{N}) \), say \( \sigma(\mathcal{N}_k) \). Then we have:

\[
\bigcup_{k \in \mathbb{N}} A_k \subset \bigcup_{k \in \mathbb{N}} \sigma(\mathcal{N}) \subset \bigcup_{\mathcal{N} \in \mathcal{P}_\mathcal{N}(E)} \sigma(\mathcal{N}).
\]  

Moreover, if if \( A \in \bigcup_{\mathcal{N} \in \mathcal{P}_\mathcal{N}(E)} \sigma(\mathcal{N}) \), \( A \) must belong to some \( \sigma(\mathcal{N}) \), say \( \sigma(\mathcal{N}_k) \). Therefore \( A^c \in \sigma(\mathcal{N}_k) \). This shows that \( \bigcup_{\mathcal{N} \in \mathcal{P}_\mathcal{N}(E)} \sigma(\mathcal{N}) \) is a \( \sigma \)-algebra. Now we show that

\[
\sigma(E) = \bigcup_{\mathcal{N} \in \mathcal{P}_\mathcal{N}(E)} \sigma(\mathcal{N}).
\]  

If \( A \in M = \sigma(E) \), \( A \) must belong to some \( \mathcal{N} \in \mathcal{P}_\mathcal{N}(E) \), say \( \mathcal{N} \). But, by the definition of \( \sigma(E) \), we have:

\[
\sigma(E) \subseteq \bigcup_{\mathcal{N} \in \mathcal{P}_\mathcal{N}(E)} \sigma(\mathcal{N}).
\]  

On the other hand, if \( \mathcal{N} \in \mathcal{P}_\mathcal{N}(E) \), then \( \sigma(\mathcal{N}) \subseteq \sigma(E) \) and

\[
\bigcup_{\mathcal{N} \in \mathcal{P}_\mathcal{N}(E)} \sigma(\mathcal{N}) \subseteq \sigma(E)
\]  

Solution of problem 1.3.11, pag. 27 If \( \mu \) is a measure, then the rest is obvious (see Theo. 1.8). Conversely, let \( \mu \) be a f.a.m. that is continuous from below. If \( A_k \) is a seq. of disjoint \( \mu \)-measurable sets, we have:

\[
\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{N=1}^{+\infty} \bigcup_{k=1}^{N} A_k\right).
\]  

Now we use the property of continuity from below:

\[
\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{N=1}^{+\infty} \bigcup_{k=1}^{N} A_k\right) = \lim_{N \to +\infty} \mu\left(\bigcup_{k=1}^{N} A_k\right) =
\]
\[
\lim_{N \to +\infty} \sum_{k=1}^{N} \mu(A_k) = \sum_{k=1}^{+\infty} \mu(A_k). \quad (10)
\]

Therefore \( \mu \) is a measure.

Now, again, if \( \mu \) is a measure, then the rest is obvious (see Theo. 1.8). Conversely, we suppose \( \mu(X) < +\infty \) and let \( \mu \) be a f.a.m. that is continuous from above. Then, If \( A_k \) is a seq. of disjoint \( \mu \)-measurable sets, we have:

\[
\bigcup_{k=1}^{\infty} A_k = X \setminus \bigcap_{k=1}^{\infty} A_k^c = X \setminus \bigcap_{N=1}^{\infty} \bigcap_{k=1}^{N} A_k^c. \quad (11)
\]

But \( \bigcap_{k=1}^{N} A_k^c \) is a decreasing seq., therefore:

\[
\mu(X) - \mu \left( \bigcap_{N=1}^{\infty} \bigcap_{k=1}^{N} A_k^c \right) = \mu(X) - \lim_{N \to +\infty} \mu \left( \bigcap_{k=1}^{N} A_k^c \right) = \mu(X) - \lim_{N \to +\infty} \mu \left( X \setminus \bigcup_{k=1}^{N} A_k^c \right) = \mu(X) - \mu(X) + \lim_{N \to +\infty} \sum_{k=1}^{+\infty} \mu(A_k) = \sum_{k=1}^{+\infty} \mu(A_k). \quad (12)
\]

Therefore \( \mu \) is a measure.

**Solution of problem 1.3.11, pag. 27** By hypothesis, there exists a seq. \( E_k \) of \( \mu \)-measurable sets such that \( \bigcup_{k=1}^{N} E_k = X, \mu(E_k) < +\infty \). It’s easy to see that we can suppose this seq. to be increasing. Now, if \( E \) is a \( \mu \)-measurable set with \( \mu(E) = +\infty \), let’ consider the seq. \( E_k \cap E \). We have:

\[
\bigcup_{k=1}^{+\infty} E \cap E_k = E \Rightarrow +\infty = \lim_{N \to +\infty} \mu(E \cap E_k). \quad (13)
\]

Therefore ther must be an index \( k_0 \) such that \( \mu(E \cap E_{k_0}) > 0 \). But \( \mu(E \cap E_{k_0}) \leq \mu(E_{k_0}) < +\infty \). Therefore \( \mu \) is semi-finite.