1. Determine whether the following statements are true or false. Give brief justification for your answer.
   (a) There exists \( f \in L^1(\mathbb{T}) \) such that \( S_N f \) does not tend to \( f \) in \( L^1 \) norm as \( N \to \infty \).
   (b) The Fourier transform is bounded from \( L^p(\mathbb{R}^d) \) to \( L^{p'}(\mathbb{R}^d) \) for every \( 1 \leq p \leq 2 \).
   (c) The Fourier transform is bounded from \( L^p(\mathbb{R}^d) \) to \( L^{p'}(\mathbb{R}^d) \) for some \( p > 2 \).

2. Recall that the total variation \( V(f, I) \) of a function \( f \) over an interval \( I = (a, b) \subseteq \mathbb{R} \) is
   \[
   V(f, I) = \sup_P \sum_{j=1}^{k} |f(t_j) - f(t_{j-1})|,
   \]
   where the supremum is taken over all partitions \( P = \{(t_0, t_1, \ldots, t_k) : a < t_0 < t_1 < \cdots < t_k < b \} \). We denote \( V(f) = V(f, \mathbb{R}) \) if \( f \) is globally defined, and say that \( f \) is of bounded variation, denoted \( f \in BV(\mathbb{R}) \), if \( V(f) < \infty \).
   If \( f \in BV(\mathbb{R}) \) and has compact support, or if \( f \in BV(\mathbb{T}) \), show that
   \[
   |\hat{f}(\xi)| \leq 2\pi V(f) |\xi|^{-1}
   \]
   for all \( \xi \neq 0 \).
   Convince yourself that the proofs should be essentially identical for the two cases, and present just one.

3. The aim of this problem is to show that if \( f \in C(\mathbb{T}) \) has bounded variation, then \( S_N f \to f \) uniformly as \( N \to \infty \). Fill in these steps to arrive at this conclusion.
   (a) Let \( g \in L^1(\mathbb{T}) \). Suppose that the Fourier coefficients of \( g \) satisfy the property that for every \( \epsilon > 0 \) there is \( \lambda = \lambda(\epsilon) > 1 \) such that
   \[
   \limsup_{n \to \infty} \sum_{n \leq |k| \leq \lambda n} |\hat{g}(k)| < \epsilon.
   \]
   Show that \( S_n(g)(t) \) converges if and only if \( \sigma_n(g)(t) \) does.
   (b) Use Problem 2 and the result in part (a) to prove the desired convergence of \( S_N f \) if \( f \in C(\mathbb{T}) \cap BV(\mathbb{T}) \).
4. We proved in class that there exists \( f \in C(T) \) whose Fourier series fails to converge uniformly, in fact, diverges to infinity at a point. Use the strategy outlined below (due to Salem) to prove a result of Pál and Bohr, which provides an interesting counterpoint: for any real-valued \( f \in C(T) \), there exists a homeomorphism, in fact a continuous strictly increasing function \( \varphi : T \rightarrow T \) with \( \varphi(0) = 0, \varphi(2\pi) = 2\pi \), such that \( S_N(f \circ \varphi) \rightarrow f \circ \varphi \) uniformly on \( T \).

(a) Argue that without loss of generality \( f \) can be chosen to have mean value zero and to vanish at the endpoints 0 and \( 2\pi \). Deduce from this that there exists \( 0 < a < 2\pi \) such that \( f(a) = 0 \).

(b) Suppose first that the point \( a \) defined in part (a) is unique. Prove the Pál-Bohr theorem under this restricted assumption. (Hint: You may use without proof the following version of the Riemann mapping theorem: Let \( \Omega \) be a domain in the plane bounded by a simple closed curve \( \gamma \). Then there exists a conformal mapping (i.e. a holomorphic bijection) of the unit disc \( \mathbb{D} = \{ |z| < 1 \} \) onto \( \Omega \). The power series representing any such mapping converges uniformly on \( \partial \mathbb{D} \) and determines a continuous bijection of \( \partial \mathbb{D} \) onto \( \gamma \).)

(c) Suppose now that \( a \) is not the only point on \((0, 2\pi)\) where \( f \) vanishes. Let \( t_1 \in I_1 = (0, a) \) and \( t_2 \in I_2 = (a, 2\pi) \) be two points such that \( |f(t_i)| = \max \{|f(t)| : t \in I_i\} \). Consider the function \( \omega(t) \) defined as follows,

\[
\omega(t) = \begin{cases} 
\max_{0 \leq s \leq t} |f(s)| + \sin\left(\frac{\pi t}{a}\right) & \text{if } 0 \leq t \leq t_1 \\
\max_{t \leq s \leq a} |f(s)| + \sin\left(\frac{\pi t}{a}\right) & \text{if } t_1 \leq t \leq a \\
-\max_{a \leq s \leq t} |f(s)| - \sin\left(\frac{\pi(t - a)}{2\pi - a}\right) & \text{if } a \leq t \leq t_2 \\
-\max_{t \leq s \leq 2\pi} |f(s)| - \sin\left(\frac{\pi(t - a)}{2\pi - a}\right) & \text{if } t_2 \leq t \leq 2\pi.
\end{cases}
\]

Show that \( \omega \) is of bounded variation and that \( f + \omega \) satisfies the hypothesis of part (b).

(d) Use part (c) and problem 3 to complete the proof of the theorem.