1. Determine whether each of the following statements is true or false. Give brief justification for your answers.
   (a) For any $p \in [1, \infty)$, the set of all functions in $L^p(\mathbb{R}^d)$ whose Fourier transforms have compact support is dense in $L^p(\mathbb{R}^d)$.
   (b) There exists $p \neq 2$ such that the Fourier transform can be extended as a bounded linear map from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$.
   (c) There exists $f \in L^\infty \cup 1 \leq p < \infty L^p$ whose Fourier transform is a function, not merely a distribution.
   (d) If $f \in C(\mathbb{T}^d)$ satisfies a Hölder condition with exponent $\alpha$, $0 < \alpha \leq 1$, then $\hat{f}(n) = O(|n|^{-\alpha})$.
   (e) There exists a sequence $\epsilon_n \to 0$ with the following property: for every $f \in C(\mathbb{T})$, $|\hat{f}(n)| < \epsilon_n$ for all sufficiently large $n$.

2. Evaluation of the Fourier transform for some examples. Establish the following identities:
   (a) For any $\xi \in \mathbb{R}$, 
      $$\left[(1 + x^2)^{-1}\right]^{-\frac{1}{2}}(\xi) = \pi e^{-|\xi|}.$$ 
   (b) For any $\xi \in \mathbb{R}$, 
      $$\left[e^{-x^2/2}\right]^{-\frac{1}{2}}(\xi) = \sqrt{2\pi}e^{-\xi^2/2}.$$ 
   (c) In $\mathbb{R}^d$, 
      $$\left[e^{-|\xi|^2}\right]^{-\frac{1}{2}}(x) = (2\pi)^d \pi^{-\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right) \left(1 + |x|^2\right)^{-\frac{d+1}{2}}.$$ 
      As part of the proof of Plancherel’s theorem presented in class, we showed that there exists an absolute constant $\beta > 0$ such that $\langle \hat{f} \rangle^\vee = \beta^{-d}f$ almost everywhere for every $d \geq 1$ and $f \in L^1(\mathbb{R}^d)$. Use any subset of the formulae above to find the value of $\beta$.
      (Hint for (c): Start by writing $e^{-|\xi|}$ as
      $$e^{-|\xi|} = \pi^{-\frac{d}{2}} \int_0^\infty e^{-\frac{|\xi|^2}{4u}} e^{-u} u^{-\frac{1}{2}} du,$$
      then evaluate the Fourier transform using (b).)

3. Recall the definition of temperate measure introduced in class.
(a) Given any temperate measure $\mu$ and multi-index $\alpha$, show that $\varphi$ defined by
\[
\langle \varphi, f \rangle := \int_{\mathbb{R}^d} \partial^\alpha f(x) \, d\mu(x), \quad f \in \mathcal{S}(\mathbb{R}^d)
\]
is a tempered distribution.

(b) Show that every element of $\mathcal{S}'(\mathbb{R}^d)$ is a finite linear combination of distributions of the form (\(*\)), for certain temperate measures $\mu$ and multi-indices $\alpha$.

4. This problem deals with some more examples of tempered distributions.
(a) Show that the principal value $1/x$ distribution, defined by
\[
\text{pv} \int f(x) x^{-1} \, dx := \lim_{\epsilon \to 0} \int_{|x| > \epsilon} f(x) x^{-1} \, dx
\]
is tempered.

(b) Show that $\varphi(\cdot)$, defined by
\[
\langle \varphi(z), f \rangle = \int_0^{\infty} x^z f(x) \, dx
\]
is a holomorphic $\mathcal{S}'$-valued function of $z$ for $\text{Re}(z) > -1$, which can be continued as a meromorphic $\mathcal{S}'$-valued function on all of $\mathbb{C}$, with only simple poles at $z = -1, -2, -3, \ldots$.

5. Poisson summation formula. The Fourier transforms for $\mathbb{R}^d$ and $\mathbb{T}^d$ are related. An elegant application of this linkage yields the Poisson summation formula, which states that for every $f \in C^\infty_c(\mathbb{R}^d)$,
\[
\sum_{n \in \mathbb{Z}^d} f(2\pi n) = (2\pi)^{-d} \sum_{n \in \mathbb{Z}^d} \hat{f}(n).
\]
Follow the steps outlined below to arrive at a proof of this formula.
(a) Given $f$, form the $(2\pi \mathbb{Z})^d$-periodic function $F(x) = \sum_{k \in \mathbb{Z}^d} f(x + 2\pi k)$. Argue that $F$ equals its Fourier series.

(b) Show that
\[
\hat{F}(n) = (2\pi)^{-d} \hat{f}(n) \quad \text{for all } n \in \mathbb{Z}^d,
\]
where $\hat{\cdot}$ denotes the $\mathbb{T}^d$ Fourier transform on the left and the $\mathbb{R}^d$ Fourier transform on the right.

(c) Set $x = 0$ and combine the previous steps to arrive at the Poisson summation formula.